

BERNSTEIN TYPE INEQUALITIES FOR RESTRICTIONS OF POLYNOMIALS TO COMPLEX SUBMANIFOLDS OF \mathbb{C}^N

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ABSTRACT. The paper studies Bernstein type inequalities for restrictions of holomorphic polynomials to graphs $\Gamma_f \subset \mathbb{C}^{n+m}$ of holomorphic maps $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$. We establish general properties of exponents in such inequalities and describe some classes of graphs admitting Bernstein type inequalities of optimal exponents and of exponents of polynomial growth.

1. FORMULATION OF MAIN RESULTS

1.1. In recent years there was a considerable interest in Bernstein, Markov and Remez type inequalities for restrictions of holomorphic polynomials to certain submanifolds of \mathbb{C}^N in connection with various problems of analysis and geometry, see, e.g., [B, BBL, BBLT, BLMT, BP, CP1, CP2, CP3, FN1, FN2, NSV, P, RY, S] and references therein. Specifically, the graph $\Gamma_f \subset \mathbb{C}^{n+m}$ of a holomorphic map $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is said to admit the *Bernstein type inequality of exponent* $\mu : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ if for each $r > 0$ there exists a nonnegative constant $C(r)$ such that for all holomorphic polynomials p on \mathbb{C}^{n+m}

$$(1.1) \quad \max_{\|z\| \leq er} |p(z, f(z))| \leq C(r)^{\mu(\deg p)} \max_{\|z\| \leq r} |p(z, f(z))|.$$

(Here $\|\cdot\|$ is the Euclidean norm on \mathbb{C}^n .)

The value $\mu(\deg p) \ln C(r)$ can be regarded as the degree of the function $p_f := p(\cdot, f(\cdot))$. In particular, inequality (1.1) implies the corresponding Markov and Remez type inequalities for functions p_f in the Euclidean balls $\{z \in \mathbb{C}^n : \|z\| \leq r\}$ with degrees of polynomials in the standard setting (see [Ma, Be, R, BG]) replaced by $c\mu(\deg p) \ln C(r)$ for an absolute constant $c > 0$, see, e.g., [B, Sect. 2], [BLMT] for details. In addition, if $n = 1$, inequality (1.1) implies the Jensen type inequality asserting that the number of zeros (counted with their multiplicities) of the function p_f in the disc $\{z \in \mathbb{C} : |z| \leq r\}$ is bounded from above by $\frac{5}{2}\mu(\deg p) \ln C(r)$, see, e.g., [VP].

It is known that Γ_f admits the Bernstein type inequality of exponent $\mu_{\text{id}}(k) := k$, $k \in \mathbb{Z}_+$, if and only if it is a complex algebraic manifold, see [S]. On the other hand, it is easy to give examples of graphs Γ_f for which the exponent μ in (1.1) must be of an arbitrarily prescribed growth, see, e.g., [BBL, p. 140].

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In this paper we begin the systematic study of general properties of exponents in Bernstein type inequalities and of some classes of graphs Γ_f admitting such inequalities of exponents of polynomial growth. Some of our proofs rely heavily upon the results of [B].

1.2. In this part we describe some properties of exponents in Bernstein type inequalities.

Recall that a subset $K \subset \mathbb{C}^n$ is called *pluripolar* if there exists a nonidentical $-\infty$ plurisubharmonic function u on \mathbb{C}^n such that $u|_K = -\infty$. (For basic results of the theory of plurisubharmonic functions see, e.g., [K].)

Theorem 1.1. (a) *For each holomorphic map $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ its graph $\Gamma_f \subset \mathbb{C}^{n+m}$ admits the Bernstein type inequality of certain exponent.*

(b) *$\Gamma_f \subset \mathbb{C}^{n+m}$ admits the Bernstein type inequality of exponent μ if and only if for each compact nonpluripolar subset $K \subset \mathbb{C}^n$ there exists a constant $C(K; r)$, $r > 0$, such that for all $p \in \mathcal{P}(\mathbb{C}^{n+m})$, the space of holomorphic polynomials on \mathbb{C}^{n+m} ,*

$$(1.2) \quad \max_{\|z\| \leq r} |p(z, f(z))| \leq C(K; r)^{\mu(\deg p)} \max_{z \in K} |p(z, f(z))|.$$

(c) *If $\Gamma_f \subset \mathbb{C}^{n+m}$ is nonalgebraic and admits the Bernstein type inequality of exponent μ , then*

$$\lim_{k \rightarrow \infty} \frac{\mu(k)}{k^{1+\frac{1}{n}}} \neq 0.$$

(d) *If $\Gamma_f \subset \mathbb{C}^{n+m}$ admits the Bernstein type inequalities of exponents μ_1 and μ_2 , then it admits such inequalities of all exponents $\mu \geq \min(\mu_1, \mu_2)$.*

(e) *If $\Gamma_f \subset \mathbb{C}^{n+m}$ admits the Bernstein type inequality of exponent μ , then each $\Gamma_{f_w} \subset \mathbb{C}^{n+m}$, $f_w(z) := f(z + w)$, $z \in \mathbb{C}^n$, $w \in \mathbb{C}^n$, admits it as well.*

(f) *If graphs $\Gamma_{f_i} \subset \mathbb{C}^{n_i+m_i}$ of holomorphic maps $f : \mathbb{C}^{n_i} \rightarrow \mathbb{C}^{m_i}$ admit Bernstein type inequalities of exponents μ_i , $i = 1, 2$, then the graph $\Gamma_{f_1 \times f_2} \subset \mathbb{C}^{n_1+n_2+m_1+m_2}$ of the map $(f_1 \times f_2)(z_1, z_2) := (f_1(z_1), f_2(z_2)) \in \mathbb{C}^{m_1+m_2}$, $z_i \in \mathbb{C}^{n_i}$, $i = 1, 2$, admits the Bernstein type inequality of exponent $\max(\mu_1, \mu_2)$.*

In turn, if $\Gamma_{f_1 \times f_2}$ admits the Bernstein type inequality of exponent μ , then each Γ_{f_i} admits it as well.

We say that functions $\mu_1, \mu_2 : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ are *equivalent* if there exists a positive real number c such that

$$\frac{1}{c} \mu_1(k) \leq \mu_2(k) \leq c \mu_1(k) \quad \text{for all } k \in \mathbb{Z}_+.$$

Let \mathcal{R} be the set of equivalence classes of functions $\mathbb{Z}_+ \rightarrow \mathbb{R}_+$. By $\langle \mu \rangle \in \mathcal{R}$ we denote the equivalence class of $\mu : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$. We introduce a partial order on \mathcal{R} writing $\langle \mu_1 \rangle \leq \langle \mu_2 \rangle$ if there exists $c > 0$ such that $\mu_1 \leq c \mu_2$. In addition, we regard \mathcal{R} as an abelian semigroup with addition $\langle \mu_1 \rangle + \langle \mu_2 \rangle := \langle \max(\mu_1, \mu_2) \rangle$ induced by the pointwise addition of functions. Clearly, if Γ_f admits the Bernstein type inequality of exponent μ , then it admits such inequality of any equivalent exponent. Therefore it is naturally to consider the set $\mathcal{E}_f \subset \mathcal{R}$ of equivalence classes of possible exponents in Bernstein type inequalities for Γ_f . Then properties (c)–(f) of the theorem can be rephrased as follows:

- (c') If $\Gamma_f \subset \mathbb{C}^{n+m}$ is nonalgebraic, then $\langle \mu_{\text{id}}^{1+\frac{1}{n}} \rangle \in \mathcal{R}$ is a lower bound of the set \mathcal{E}_f .
- (d') \mathcal{E}_f is a partially ordered subsemigroup of $(\mathcal{R}, \leq, +)$ and every two elements of \mathcal{E}_f have unique infimum and supremum (i.e. \mathcal{E}_f is a *lattice*).
- (e') \mathcal{E}_f coincides with \mathcal{E}_{f_w} for all $w \in \mathbb{C}^n$.
- (f') $\mathcal{E}_{f_1 \times f_2} = \mathcal{E}_{f_1} + \mathcal{E}_{f_2}$.

We say that a function $\mu_o : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ is *optimal* for Γ_f if $\langle \mu_o \rangle$ is the minimal element of \mathcal{E}_f (in other words, Γ_f admits the Bernstein type inequality of exponent μ_o and does not admit such inequality of an exponent μ such that $\mu \leq c\mu_o$ for some $c > 0$ and $\overline{\lim}_{k \rightarrow \infty} \frac{\mu_o(k)}{\mu(k)} = \infty$.) Since \mathcal{E}_f is a lattice, the minimal element $\langle \mu_0 \rangle$ of \mathcal{E}_f is also the least element of \mathcal{E}_f , i.e. $\langle \mu_0 \rangle \leq \langle \mu \rangle$ for all $\langle \mu \rangle \in \mathcal{E}_f$. Moreover, in this case $\mathcal{E}_f = \langle \mu_0 \rangle + \mathcal{R}$.

For instance, μ_{id} is optimal for an algebraic Γ_f . Below we give some other examples of Γ_f allowing optimal exponents. The problem of existence of optimal exponents for generic Γ_f is open.

Let $K \subset \mathbb{C}^n$ be a nonpluripolar compact set. For a function $\mu : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ we set

$$u_{K,\mu}^k(z; f) := \sup \left\{ \frac{\ln |p_f(z)|}{\max(1, \mu(k))} : p \in \mathcal{P}(\mathbb{C}^{n+m}), \deg p = k, \sup_K |p_f| = 1 \right\}, \quad z \in \mathbb{C}^n,$$

$$u_K^k(r; f) := \max(1, \mu(k)) \max_{\|z\| \leq r} u_{K,\mu}^k(z; f), \quad r > 0, \quad k \in \mathbb{Z}_+.$$

Approximating polynomials of a given degree by those of a larger one (cf. (3.21) below), one obtains that for each $r > 0$ the sequence $u_K^k(r; f)$, $k \in \mathbb{Z}_+$, is nondecreasing. Also, due to the maximum principle for plurisubharmonic functions classes $\langle u_K^k(r; f) \rangle \in \mathcal{R}$, $r > 0$, form a subsemigroup and a chain \mathcal{U}_f^K . By definition, each element of \mathcal{E}_f is an upper bound of \mathcal{U}_f^K .

Theorem 1.2. (a) *Each $u_{K,\mu}^k$ is a nonnegative continuous plurisubharmonic function on \mathbb{C}^n equals 0 on K .*
 (b) *Graph Γ_f admits the Bernstein type inequality of exponent μ if and only if the (Lebesgue measurable) function*

$$u_{K,\mu}(z; f) := \overline{\lim}_{k \rightarrow \infty} u_{K,\mu}^k(z; f), \quad z \in \mathbb{C}^n,$$

is locally bounded from above.

(c) *An exponent μ in the Bernstein type inequality for Γ_f is optimal if and only if for each subsequence $\bar{k} = \{k_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ the function*

$$u_{K,\mu;\bar{k}}(z; f) := \overline{\lim}_{j \rightarrow \infty} u_{K,\mu}^{k_j}(z; f), \quad z \in \mathbb{C}^n,$$

is not identically 0.

(d) *An exponent μ in the Bernstein type inequality for Γ_f is optimal if and only if $\langle \mu \rangle \in \mathcal{U}_f^K$. In this case $\langle \mu \rangle$ is the maximal element of \mathcal{U}_f^K .*

Remark 1.3. (1) Theorem 1.1 (f) implies that if functions $\mu_i : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ are optimal for Γ_{f_i} , $i = 1, 2$, then the function $\max(\mu_1, \mu_2)$ is optimal for $\Gamma_{f_1 \times f_2}$.

(2) For a nonpolynomial entire function $f : \mathbb{C} \rightarrow \mathbb{C}^m$ and $p \in \mathcal{P}(\mathbb{C}^{m+1})$ by $n_{p_f}(r)$ we denote the number of zeros (counted with their multiplicities) of the univariate holomorphic function p_f in the closed disk $\bar{\mathbb{D}}_r := \{z \in \mathbb{C} : |z| \leq r\}$. (We set $n_{p_f} = -\infty$ if $p_f = 0$.) Let

$$N^k(r; f) := \sup\{n_{p_f}(r) : p \in \mathcal{P}(\mathbb{C}^{m+1}), \deg p \leq k\}.$$

The integer-valued function $N^k(\cdot; f)$ is nonnegative locally bounded from above and satisfies for all $r > 1$ (see [CP1, Cor. 2.3]),

$$\frac{1}{\ln r + 16} u_{\bar{\mathbb{D}}_1}^k\left(\frac{r}{3}\right) \leq N^k(r; f) \leq 2u_{\bar{\mathbb{D}}_1}^k(3r).$$

Thus, the classes $\langle N(r; f) \rangle \in \mathcal{R}$, $r > 0$, form a subsemigroup and a chain \mathcal{N}_f such that

$$\langle u_{\bar{\mathbb{D}}_1}\left(\frac{r}{3}\right) \rangle \leq \langle N(r; f) \rangle \leq \langle u_{\bar{\mathbb{D}}_1}(3r) \rangle \quad \text{for all } r > 1.$$

In particular, Theorem 1.2 (b),(d) implies that $\Gamma_f \subset \mathbb{C}^{m+1}$ admits the Bernstein type inequality of an exponent μ if and only if $\langle \mu \rangle \in \mathcal{R}$ is an upper bound of \mathcal{N}_f . In addition, such μ is optimal for Γ_f if and only if $\langle \mu \rangle \in \mathcal{N}_f$. In this case, $\langle \mu \rangle \in \mathcal{R}$ is the maximal element of \mathcal{N}_f so that as an optimal exponent one can take, e.g., the function $N(r_0; f) : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ for a sufficiently large r_0 .

1.3. In this section we describe some classes of graphs Γ_f admitting Bernstein type inequalities of exponents of polynomial growth.

First, we show that power functions $\mu_{\text{id}}^d(k) := k^d$, $k \in \mathbb{Z}_+$, $d \in \mathbb{N}$, are optimal exponents in Bernstein type inequalities on some graphs Γ_f .

In what follows, for holomorphic maps $f_j : \mathbb{C}^{n_j} \rightarrow \mathbb{C}^{m_j}$, $1 \leq j \leq l$, by $f_1 \times \cdots \times f_l : \mathbb{C}^{n_1+\cdots+n_l} \rightarrow \mathbb{C}^{m_1+\cdots+m_l}$ we denote a map given by the formula

$$(1.3) \quad (f_1 \times \cdots \times f_l)(z_1, \dots, z_l) := (f_1(z_1), \dots, f_l(z_l)), \quad z_j \in \mathbb{C}^{n_j}, \quad 1 \leq j \leq l.$$

A map $f : \mathbb{C} \rightarrow \mathbb{C}^m$ is said to be *exponential of maximal transcendence degree* if there are linearly independent over \mathbb{Q} complex numbers $\alpha_1, \dots, \alpha_m$ such that

$$(1.4) \quad f(z) := (e^{\alpha_1 z}, \dots, e^{\alpha_m z}), \quad z \in \mathbb{C}.$$

(In this case the coordinates of f are algebraically independent over the field of rational functions on \mathbb{C} and so the *Zariski closure* of $\Gamma_f \subset \mathbb{C}^{m+1}$ coincides with \mathbb{C}^{m+1} .)

Let $f_j : \mathbb{C} \rightarrow \mathbb{C}^{m_j}$, $1 \leq j \leq l$, be exponential maps of maximal transcendence degrees and P, Q be holomorphic polynomial automorphisms of \mathbb{C}^l and $\mathbb{C}^{m_1+\cdots+m_l}$, respectively. We set

$$(1.5) \quad \bar{m} := \max_{1 \leq j \leq l} m_j \quad \text{and} \quad F_{P,Q} := Q \circ (f_1 \times \cdots \times f_l) \circ P.$$

(The coordinates of the map $F_{P,Q} : \mathbb{C}^l \rightarrow \mathbb{C}^{m_1+\cdots+m_l}$ are functions of the form $\sum_{j=1}^J p_j e^{q_j}$, $p_j, q_j \in \mathcal{P}(\mathbb{C}^l)$, $1 \leq j \leq J$, called the *generalized exponential polynomials* on \mathbb{C}^l .)

Theorem 1.4. *Graph $\Gamma_{F_{P,Q}} \subset \mathbb{C}^{l+m_1+\dots+m_l}$ admits the Bernstein type inequality of optimal exponent $\mu_{\text{id}}^{\bar{m}+1}$.*

Our next result reveals the basic property of Bernstein type inequalities on the graphs of nonpolynomial entire functions.

Theorem 1.5. *Let f be a nonpolynomial entire function on \mathbb{C}^n . Then its graph $\Gamma_f \subset \mathbb{C}^{n+1}$ admits the Bernstein type inequality of an exponent μ such that*

$$(1.6) \quad 1 + \frac{1}{n} \leq \varliminf_{k \rightarrow \infty} \frac{\ln \mu(k)}{\ln k} \leq 2.$$

It is unclear whether this result is sharp as currently there are no examples of graphs $\Gamma_f \subset \mathbb{C}^n$, $n \geq 2$, admitting Bernstein type inequalities of exponents μ for which the corresponding limit in (1.6) is strictly less than two.

In our subsequent formulations we use the following definitions and notation.

By $\mathbb{B}_r^n \subset \mathbb{C}^n$ we denote the open Euclidean ball of radius r centered at 0; we set $\mathbb{B}^n := \mathbb{B}_1^n$, $\mathbb{D}_r := \mathbb{B}_r^1$ and $\mathbb{D} := \mathbb{B}^1$.

For a continuous function $f : \mathbb{B}_r^n \rightarrow \mathbb{C}$ we define

$$M_f(r) := \sup_{\mathbb{B}_r^n} |f|, \quad m_f(r) := \ln M_f(r).$$

Next, recall that an entire function f on \mathbb{C}^n is of order $\rho_f \geq 0$ if

$$\rho_f = \overline{\lim}_{r \rightarrow \infty} \frac{\ln m_f(r)}{\ln r}.$$

If $\rho_f < \infty$, then f is called of finite order.

For a nonconstant entire function f on \mathbb{C}^n of order ρ_f we set

$$\phi_f(t) := m_f(e^t), \quad t \in \mathbb{R}.$$

Then ϕ_f is a convex increasing function.

By \mathcal{C} we denote the class of nonpolynomial entire functions f satisfying one of the following conditions:

(I) If $\rho_f < \infty$,

$$(1.7) \quad \overline{\lim}_{t \rightarrow \infty} \frac{\phi_f(t+1) - \phi_f(t)}{\phi_f(t) - \phi_f(t-1)} < \infty.$$

(II) If $\rho_f = \infty$,

$$(1.8) \quad \lim_{t \rightarrow \infty} t^2 \left(\frac{1}{\ln \psi(t)} \right)' = 0,$$

where $\psi \in C(\mathbb{R})$ is a convex increasing function such that

$$\lim_{t \rightarrow \infty} \frac{\ln \psi(t)}{\ln \phi_f(t)} = 1.$$

Remark 1.6. (1) Each convex function is differentiable at all but at most countably many points. Thus the limit in (1.8) is taken over the domain of ψ' .

(2) Conditions (1.7) and (1.8) are complimentary to each other (i.e. there are no entire functions satisfying both of these conditions).

Let $f_j : \mathbb{C}^{n_j} \rightarrow \mathbb{C}$, $1 \leq j \leq m$, be functions of class \mathcal{C} and P, Q be holomorphic polynomial automorphisms of $\mathbb{C}^{n_1+\dots+n_m}$ and \mathbb{C}^m , respectively.

Theorem 1.7. *The graph $\Gamma_{F_{P,Q}} \subset \mathbb{C}^{n_1+\dots+n_m+m}$ of the map $F_{P,Q} := Q \circ (f_1 \times \dots \times f_m) \circ P : \mathbb{C}^{n_1+\dots+n_m} \rightarrow \mathbb{C}^m$ admits the Bernstein type inequality of exponent $\mu(k) := k^{2+\varepsilon(k)}$, $k \in \mathbb{Z}_+$, where $\varepsilon = 0$ if all $\rho_{f_j} < \infty$ and $\varepsilon \geq 0$ decreases to 0 if one of $\rho_{f_j} = \infty$.*

Moreover, if all $n_j = 1$ and all $\rho_{f_j} < \infty$, then μ_{id}^2 is an optimal exponent for $\Gamma_{F_{P,Q}}$.

Our next result describes some class of curves $\Gamma_f \subset \mathbb{C}^{m+1}$ admitting Bernstein type inequalities of exponents of polynomial growth. Up till now, for $m \geq 2$ the only known examples of such curves were graphs of holomorphic maps $f : \mathbb{C} \rightarrow \mathbb{C}^m$ with coordinates being exponents of polynomials. As follows from the results established in [BBL] graphs of such maps admit Bernstein type inequalities of exponents μ_{id}^{3m+3} .

Theorem 1.8. *Suppose that nonpolynomial entire functions $f_j : \mathbb{C} \rightarrow \mathbb{C}$, $1 \leq j \leq m$, of class \mathcal{C} are such that $\rho_{f_1} \leq \dots \leq \rho_{f_m}$ and*

$$(1.9) \quad \lim_{r \rightarrow \infty} \frac{m_{f_j}(r) - m_{f_j}\left(\frac{r}{e}\right)}{\sqrt{m_{f_{j+1}}(r) - m_{f_{j+1}}\left(\frac{r}{e}\right)}} = 0 \quad \text{if } \rho_{f_{j+1}} < \infty, \quad j \in \{1, \dots, m-1\},$$

and

$$(1.10) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\ln m_{f_j}(r)}{\ln m_{f_{j+1}}\left(\frac{r}{e}\right)} < \frac{1}{2} \quad \text{if } \rho_{f_j} = \infty, \quad j \in \{1, \dots, m-1\}.$$

Then for $f = (f_1, \dots, f_m) : \mathbb{C} \rightarrow \mathbb{C}^m$ its graph $\Gamma_f \subset \mathbb{C}^{m+1}$ admits the Bernstein type inequality of exponent $\mu(k) := k^{2^m+\varepsilon(k)}$, $k \in \mathbb{Z}_+$, for some $\varepsilon : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ decreasing to zero. Here $\varepsilon = 0$ if all $\rho_{f_j} < \infty$.

Remark 1.9. (1) The Zariski closure of Γ_f is \mathbb{C}^{m+1} , (i.e. each holomorphic polynomial vanishing on Γ_f is zero), cf. Remark 6.6 in section 6.3.

(2) According to the Jensen type inequality, see [VP, Lm. 1], for each $p \in \mathcal{P}(\mathbb{C}^{m+1})$ the number of zeros (counted with their multiplicities) of the function $p_f(z) := p(z, f_1(z), \dots, f_m(z))$, $z \in \mathbb{C}$, in the closed disk $\bar{\mathbb{D}}_r$, is bounded from above by $C(r)(\deg p)^{2^m+1}$ for some positive constant $C(r)$, $r > 0$.

(3) Let $f^j = (f_1^j, \dots, f_{m_j}^j) : \mathbb{C} \rightarrow \mathbb{C}^{m_j}$, $m_j \in \mathbb{N}$, $1 \leq j \leq l$, be holomorphic maps satisfying conditions of Theorem 1.8. Let P and Q be holomorphic polynomial automorphisms of \mathbb{C}^l and $\mathbb{C}^{m_1+\dots+m_l}$, respectively. We set

$$F_{P,Q} := Q \circ (f^1 \times \dots \times f^l) \circ P : \mathbb{C}^l \rightarrow \mathbb{C}^{m_1+\dots+m_l} \quad \text{and} \quad \bar{m} := \max_{1 \leq j \leq l} m_j.$$

Then the graph $\Gamma_{F_{P,Q}} \subset \mathbb{C}^{l+m_1+\dots+m_l}$ of $F_{P,Q}$ satisfies the Bernstein type inequality of exponent $\mu(k) := k^{2^m+\varepsilon(k)}$, $k \in \mathbb{Z}_+$, for some $\varepsilon : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ decreasing to zero. Here $\varepsilon = 0$ if all $\rho_{f_i^j} < \infty$.

The proof of this result follows from Theorems 1.8 and 1.1 (f) by means of the arguments of the proof of Theorem 1.7 in section 6.2 (cf. also Lemma 4.1).

We illustrate the theorem by a simple example (see section 1.4 for other examples).

Example 1.10. By $e^{\circ p} : \mathbb{C} \rightarrow \mathbb{C}$, $p \in \mathbb{N}$, we denote the p times composition of the exponential function with itself. For some $m_1, \dots, m_l \in \mathbb{N}$, we set $m := m_1 + \dots + m_l$. Consider the following univariate entire functions f_1, \dots, f_m .

If $1 \leq j \leq m_1$, then $f_j(z) = e^{z^{n_j}}$, $z \in \mathbb{C}$, $n_j \in \mathbb{N}$, where

$$\frac{n_{j+1}}{n_j} > 2 \quad \text{for all } 1 \leq j \leq m_1 - 1.$$

If $m_1 + 1 \leq j \leq m_2$, then $f_j(z) = e^{\circ 2}(n_j z)$, $z \in \mathbb{C}$, $n_j > 0$, where

$$\frac{n_{j+1}}{n_j} > 2e \quad \text{for all } m_1 + 1 \leq j \leq m_2 - 1.$$

If $m_{k-1} \leq j \leq m_k$ for $3 \leq k \leq l$, then $f_j(z) = e^{\circ k}(n_j z)$, $z \in \mathbb{C}$, $n_j > 0$, where

$$\frac{n_{j+1}}{n_j} > e \quad \text{for all } m_{k-1} + 1 \leq j \leq m_k - 1.$$

One easily checks that

$$m_{f_j}(r) = \ln f_j(r) \quad \text{for all } 1 \leq j \leq m,$$

and that all $f_j \in \mathcal{C}$ and satisfy the hypotheses of Theorem 1.8. Hence, for $f = (f_1, \dots, f_m) : \mathbb{C} \rightarrow \mathbb{C}^m$ its graph $\Gamma_f \subset \mathbb{C}^{m+1}$ admits the Bernstein type inequality of exponent $\mu(k) := k^{2^m+\varepsilon(k)}$, $k \in \mathbb{Z}_+$, for some $\varepsilon : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ decreasing to zero.

1.4. In this section we formulate some properties of functions of class \mathcal{C} .

The first three properties follow straightforwardly from the definition of class \mathcal{C} .

Proposition 1.11. (1) *If $f \in \mathcal{C}$ and g is an entire function such that*

$$0 < \varliminf_{r \rightarrow \infty} \frac{M_g(r)}{M_f(r)} \leq \overline{\lim}_{r \rightarrow \infty} \frac{M_g(r)}{M_f(r)} < \infty,$$

then $g \in \mathcal{C}$ as well.

(2) *If $f \in \mathcal{C}$, $\rho_f = \infty$, and g is an entire function such that*

$$0 < \varliminf_{r \rightarrow \infty} \frac{m_g(r)}{m_f(r)} \leq \overline{\lim}_{r \rightarrow \infty} \frac{m_g(r)}{m_f(r)} < \infty,$$

then $g \in \mathcal{C}$ as well.

(3) *If $f \in \mathcal{C}$, then $f^n \in \mathcal{C}$ for all $n \in \mathbb{N}$.*

Properties (1) and (3) imply that if $f \in \mathcal{C}$ and $p \in \mathcal{P}(\mathbb{C})$ is a nonconstant polynomial, then $p \circ f \in \mathcal{C}$.

A function $\rho \in C^1(0, \infty)$ satisfying conditions

$$\lim_{r \rightarrow \infty} \rho(r) = \rho_f \quad \text{and} \quad \lim_{r \rightarrow \infty} r \rho'(r) \ln r = 0$$

is called the *proximate order* of an entire function f if

$$\overline{\lim}_{r \rightarrow \infty} \frac{m_f(r)}{r^{\rho(r)}} =: \sigma_f \in (0, \infty).$$

It is well known that an entire function of finite order has a proximate order, see, e.g. [L, Th. I.16].

Proposition 1.12. *If an entire function of finite positive order f has a proximate order ρ satisfying condition*

$$\underline{\lim}_{r \rightarrow \infty} \frac{m_f(r)}{r^{\rho(r)}} > 0,$$

then $f \in \mathcal{C}$.

Example 1.13. Let

$$f = \sum_{j=1}^l p_j e^{q_j}, \quad p_j, q_j \in \mathcal{P}(\mathbb{C}^n), \quad 1 \leq j \leq l,$$

be the generalized exponential polynomial. (We assume that $f \notin \mathcal{P}(\mathbb{C}^n)$.) Let \hat{q}_j denote the homogeneous component of degree $\deg q_j$ of q_j (i.e. $\deg(q_j - \hat{q}_j) < \deg q_j$). Then

$$\rho_f = \max_{1 \leq j \leq l} \deg q_j \quad \text{and} \quad \sigma_f = \lim_{r \rightarrow \infty} \frac{m_f(r)}{r^{\rho_f}} = \max_{j: \deg q_j = \rho_f} \left\{ \sup_{\mathbb{B}^n} |\hat{q}_j| \right\}.$$

Thus, due to Proposition 1.12, $f \in \mathcal{C}$.

Further, if g is an entire function such that $\overline{\lim}_{r \rightarrow \infty} \frac{m_g(r)}{r^{\rho_f}} < \sigma_f$, then

$$\overline{\lim}_{r \rightarrow \infty} \frac{M_g(r)}{M_f(r)} < 1.$$

Hence, by Proposition 1.11 (1), $f + g \in \mathcal{C}$ as well.

To formulate other properties consider the subclass of \mathcal{C} of nonpolynomial entire functions satisfying condition

$$(1.11) \quad \lim_{t \rightarrow \infty} \frac{\phi_f(t)}{t^2} = \infty.$$

It is easily seen that each $f \in \mathcal{C}$ with $\rho_f = \infty$ satisfies (1.11), see Lemma 7.2 below.

Proposition 1.14. *If $f \in \mathcal{C}$ satisfies condition (1.11), then functions $e^f, \sin f, \cos f \in \mathcal{C}$. In addition, if f is univariate, then its derivative and every antiderivative are of class \mathcal{C} and satisfy (1.11).*

Remark 1.15. It is worth noting that if f is a nonpolynomial entire function, then functions $e^f, \sin f, \cos f$ are of infinite order. Thus under the hypothesis of the theorem they satisfy condition (1.8).

Example 1.16. Let f and g be as in Example 1.13. Then ϕ_{f+g} is equivalent to ϕ_f . Moreover, $\rho_f \geq 1$ and

$$\sigma_f = \lim_{t \rightarrow \infty} \frac{\phi_f(t)}{e^{\rho_f t}} \in (0, \infty).$$

Hence, $f + g$ satisfies condition (1.11). In particular, due to Proposition 1.14, functions $e^{f+g}, \sin(f+g), \cos(f+g) \in \mathcal{C}$.

To present more explicit examples of entire functions of class \mathcal{C} , we describe a subclass of univariate functions in \mathcal{C} satisfying condition (1.11) in terms of the coefficients of their Taylor expansions at $0 \in \mathbb{C}$.

Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a continuous increasing function satisfying conditions

$$(1.12) \quad \lim_{t \rightarrow \infty} \frac{h(t)}{t} = \infty \quad \text{and} \quad \overline{\lim}_{t \rightarrow \infty} \frac{h(t+1)}{h(t)} < \infty.$$

Theorem 1.17. *Let*

$$f(z) := \sum_{j=0}^{\infty} c_j z^j, \quad z \in \mathbb{C},$$

be a holomorphic function in a neighbourhood of $0 \in \mathbb{C}$ such that

$$\ln |c_j| = - \int_0^j h^{-1}(s) ds, \quad j \in \mathbb{Z}_+.$$

Then f is an entire function of finite order of class \mathcal{C} satisfying condition (1.11). Moreover,

$$\rho_f = \overline{\lim}_{t \rightarrow \infty} \frac{\ln h(t)}{t}.$$

By f_h we denote an entire function constructed by means of a function h satisfying conditions (1.12) as in the above result.

Remark 1.18. In section 8.1 we show that for each $c > \rho_{f_h}$ there exists a number t_c such that for all $t \geq t_c$

$$\int_{h^{-1}(0)}^t h(s) ds - t \leq \phi_{f_h}(t) \leq ct + \int_{h^{-1}(0)}^t h(s) ds,$$

see [L, Ch.I.2], (8.74), (8.76).

The following result allows to construct new functions of class \mathcal{C} by means of functions of the form f_h . In its formulation, by $\omega_t(\cdot; g)$, $t \geq 0$, we denote the modulus of continuity of a function $g \in C(a, \infty)$, $a \leq 0$, restricted to the interval $[0, t]$.

Proposition 1.19. *Let $h_1, h_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$ be continuous increasing functions satisfying conditions (1.12) such that*

$$\varliminf_{t \rightarrow \infty} (h_1(t) - h_2(t)) > \varliminf_{t \rightarrow \infty} \frac{\ln h_1(t)}{t} + \varliminf_{t \rightarrow \infty} \frac{\omega_t(1; h_1^{-1})}{h_1^{-1}(t)}.$$

Then all entire functions of the form $c_1 f_{h_1} + c_2 f_{h_2}$, $(c_1, c_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, are of class \mathcal{C} .

Since h^{-1} is an increasing function, the second term on the right-hand side of the above inequality is bounded from above by one.

Example 1.20. (a) Let $h(t) = (\frac{t}{\alpha})^{\frac{1}{\alpha-1}}$, $\alpha \in (1, 2)$. Then, due to Theorem 1.17, $f_h(z) := \sum_{j=0}^{\infty} c_j z^j$, $z \in \mathbb{C}$, where

$$\ln |c_j| = - \int_0^j \alpha s^{\alpha-1} ds = -j^\alpha, \quad j \in \mathbb{Z}_+,$$

is a nonpolynomial entire function of order zero of class \mathcal{C} .

Observe that

$$\omega_h := \varliminf_{t \rightarrow \infty} \frac{\omega_t(1; h^{-1})}{h^{-1}(t)} \leq \varliminf_{t \rightarrow \infty} \frac{\sup_{s \in [0, t]} (\alpha s^{\alpha-1})'}{\alpha t^{\alpha-1}} = \varliminf_{t \rightarrow \infty} \frac{\alpha - 1}{t} = 0.$$

Thus by Proposition 1.19 for each $\tilde{h} \in C(\mathbb{R}_+)$ satisfying conditions (1.12) and

$$\varliminf_{t \rightarrow \infty} (h(t) - \tilde{h}(t)) > 0,$$

all functions of the form $c_1 f_{h_1} + c_2 f_{h_2}$, $(c_1, c_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, are of class \mathcal{C} . In particular, this is true for $\tilde{h} = h - c$, $c > 0$. In this case $f_{\tilde{h}}(z) = \sum_{j=0}^{\infty} \tilde{c}_j z^j$, $z \in \mathbb{C}$, is such that

$$\ln |\tilde{c}_j| = -(j + c)^\alpha + c^\alpha, \quad j \in \mathbb{Z}_+.$$

Thus, all entire functions f of the form

$$f(z) = \sum_{j=0}^{\infty} (c_1 e^{-j^\alpha + \sqrt{-1} \theta_j} + c_2 e^{-(j+c)^\alpha + \sqrt{-1} \tilde{\theta}_j}) z^j, \quad \theta_j, \tilde{\theta}_j \in \mathbb{R}, \quad j \in \mathbb{Z}_+, \quad (c_1, c_2) \in \mathbb{C}^2 \setminus \{(0, 0)\},$$

are of class \mathcal{C} .

(b) Let $h(t) = e^{\alpha t - 1} - 1$, $\alpha > 0$. Then Theorem 1.17 implies that $f_h(z) := \sum_{j=0}^{\infty} c_j z^j$, $z \in \mathbb{C}$, where

$$\ln |c_j| = - \int_0^j \frac{\ln(s+1) + 1}{\alpha} ds = - \frac{(j+1) \ln(j+1)}{\alpha}, \quad j \in \mathbb{Z}_+,$$

is a nonpolynomial entire function of order α of class \mathcal{C} .

Next, in this case

$$\omega_h := \varliminf_{t \rightarrow \infty} \frac{\omega_t(1; h^{-1})}{h^{-1}(t)} \leq \varliminf_{t \rightarrow \infty} \frac{\sup_{s \in [0, t]} (\ln(s+1))'}{\ln(t+1)} = 0.$$

Thus by Proposition 1.19 for each $\tilde{h} \in C(\mathbb{R}_+)$ satisfying conditions (1.12) and

$$\lim_{t \rightarrow \infty} (h(t) - \tilde{h}(t)) > \alpha,$$

all functions of the form $c_1 f_{h_1} + c_2 f_{h_2}$, $(c_1, c_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, are of class \mathcal{C} . For instance, this is true for $\tilde{h} = h - c$, $c > \alpha$. Then $f_{\tilde{h}}(z) = \sum_{j=0}^{\infty} \tilde{c}_j z^j$, $z \in \mathbb{C}$, is such that

$$\ln |\tilde{c}_j| = -\frac{(j+1+c) \ln(j+1+c)}{\alpha} + \frac{(c+1) \ln(c+1)}{\alpha}, \quad j \in \mathbb{Z}_+.$$

Thus, all entire functions f of the form

$$f(z) = \sum_{j=0}^{\infty} \left(\frac{c_1 e^{\sqrt{-1} \theta_j}}{(j+1)^{\frac{j+1}{\alpha}}} + \frac{c_2 e^{\sqrt{-1} \tilde{\theta}_j}}{(j+1+c)^{\frac{j+1+c}{\alpha}}} \right) z^j, \quad \theta_j, \tilde{\theta}_j \in \mathbb{R}, \quad j \in \mathbb{Z}_+, \quad (c_1, c_2) \in \mathbb{C}^2 \setminus \{(0, 0)\},$$

are of class \mathcal{C} .

Now, as the corollary of Theorems 1.17 and 1.8 we obtain:

Corollary 1.21. *Suppose $h_1, \dots, h_m \in C(\mathbb{R}_+)$ are increasing functions satisfying condition (1.12) such that for some integer $1 \leq l \leq m$*

$$(1.13) \quad \lim_{t \rightarrow \infty} \frac{h_j(t)}{\sqrt{h_{j+1}(t)}} = 0 \quad \text{for all } 1 \leq j \leq l-1$$

and

$$(1.14) \quad \overline{\lim}_{t \rightarrow \infty} \frac{\int_M^t h_j(s) ds}{\int_M^{t-1} h_{j+1}(s) ds} < \frac{1}{2} \quad \text{for all } l+1 \leq j \leq m-1,$$

where

$$M := \max_{l+1 \leq j \leq m} \{h_j^{-1}(0)\}.$$

Then entire functions $f_{h_1}, \dots, f_{h_l}, e^{f_{h_{l+1}}}, \dots, e^{f_{h_m}}$ satisfy the hypotheses of Theorem 1.8. Therefore for $f = (f_{h_1}, \dots, f_{h_l}, e^{f_{h_{l+1}}}, \dots, e^{f_{h_m}}) : \mathbb{C} \rightarrow \mathbb{C}^m$ its graph $\Gamma_f \subset \mathbb{C}^{m+1}$ admits the Bernstein type inequality of exponent $\mu(k) := k^{2^m + \varepsilon(k)}$, $k \in \mathbb{Z}_+$, for some $\varepsilon : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ decreasing to zero. Here $\varepsilon = 0$ if $l = m$.

Remark 1.22. Note that condition

$$\overline{\lim}_{t \rightarrow \infty} \frac{h_j(t+1)}{h_{j+1}(t)} < \frac{1}{2} \quad \text{for all } l+1 \leq j \leq m-1$$

implies (1.14).

Example 1.23. Let $l_1 < l < m_1 < m$ be some natural numbers. Suppose that $h_j(t) = \left(\frac{t}{\alpha_j}\right)^{\frac{1}{\alpha_j-1}}$, $\alpha_j \in (1, 2)$, for $1 \leq j \leq m_1$ and $m_2 + 1 \leq j \leq m_3$, and $h_j(t) = e^{\alpha_j t - 1} - 1$, $\alpha_j > 0$, for $m_1 + 1 \leq j \leq l \leq m_2$ and $m_3 + 1 \leq j \leq m_4$, where

- (i) $2\alpha_{j+1} < \alpha_j + 1$ for all $1 \leq j \leq l_1$; (ii) $\alpha_j < 2\alpha_{j+1}$ for all $l_1 + 1 \leq j \leq l$;
- (iii) $\alpha_{j+1} < \alpha_j$ for all $l + 1 \leq j \leq m_1$; (iv) $\alpha_j < \alpha_{j+1}$ for all $m_1 + 1 \leq j \leq m$.

Then entire functions $f_{h_1}, \dots, f_{h_l}, e^{f_{h_{l+1}}}, \dots, e^{f_{h_m}}$ satisfy assumptions of Corollary 1.21.

2. PROOF OF THEOREM 1.1

For basic facts of complex algebraic geometry, see, e.g., the book [M].

(a) Without loss of generality we may assume that Γ_f is nonalgebraic. Let $\mathcal{I}_f \subset \mathcal{P}(\mathbb{C}^{n+m})$ be the ideal of holomorphic polynomials vanishing on Γ_f and $\mathcal{Z}_f \subset \mathbb{C}^{n+m}$ be the set of zeros of \mathcal{I}_f . Let X_f be the irreducible component of \mathcal{Z}_f containing Γ_f . Then X_f is the complex algebraic subvariety of \mathbb{C}^{n+m} of pure dimension $l \geq n + 1$. Let $U \Subset X_f$ be a relatively compact open subset such that

$$U \cap \Gamma_f = \{(z, f(z)) \in \Gamma_f : z \in \mathbb{B}^n\}.$$

Since U is a nonpluripolar subset of \mathcal{Z}_f , [S, Th. 2.2] implies that for each $r \geq 1$, there exists a constant $A(r)$ such that for all $p \in \mathcal{P}(\mathbb{C}^{n+m})$,

$$(2.15) \quad \sup_{\mathbb{B}_{re}^n} |p_f| \leq A(r)^{\deg p} \sup_U |p|, \quad \text{where } p_f := p(\cdot, f(\cdot)).$$

Next, we prove

Lemma 2.1. *For each $k \in \mathbb{Z}_+$ there exists a positive constant $c(k)$ such that for all $p \in \mathcal{P}(\mathbb{C}^{n+m})$ with $\deg p \leq k$,*

$$\sup_U |p| \leq c(k) \sup_{U \cap \Gamma_f} |p|.$$

Proof. Assume, on the contrary, that the statement is wrong for some $k_0 \in \mathbb{Z}_+$. Then there exists the sequence of polynomials $\{p_i\}_{i \in \mathbb{N}} \subset \mathcal{P}(\mathbb{C}^{n+m})$ of degrees $\leq k_0$ such that

$$\sup_U |p_i| = 1 \quad \text{for all } i \in \mathbb{N} \quad \text{and} \quad \lim_{i \rightarrow \infty} \sup_{U \cap \Gamma_f} |p_i| = 0.$$

Due to [S, Th. 2.2] the sequence $\{p_i\}_{i \in \mathbb{N}}$ is uniformly bounded on each compact subset of X_f (cf. (2.15)). Thus, due to the Montel theorem, $\{p_i\}_{i \in \mathbb{N}}$ contains a subsequence uniformly converging on compact subsets of X_f to a function $g \in C(X_f)$ holomorphic outside of the set of singular points of X_f and such that $\sup_U |g| = 1$ and $g|_{\Gamma_f} = 0$. By definition, g is a regular function on the affine algebraic variety X_f of pure dimension l . Hence, there exist polynomials $q_1, \dots, q_s \in \mathcal{P}(\mathbb{C}^{n+m})$, where $q_s|_{X_f} \not\equiv 0$, such that

$$(2.16) \quad g^s(x) + q_1(x)g^{s-1}(x) + \dots + q_s(x) = 0 \quad \text{for all } x \in X_f.$$

Equation (2.16) and the fact that $g|_{\Gamma_f} = 0$ imply that $q_s = 0$ on Γ_f . Therefore $q_s|_{X_f} = 0$ by the definition of X_f , a contradiction proving the lemma. \square

We set

$$C(r) = eA(r), \quad r \geq 1, \quad \mu(k) := \lfloor \max\{\ln c(\deg p), \deg p\} \rfloor + 1, \quad k \in \mathbb{Z}_+.$$

Then using the lemma and equation (2.15) we obtain, for each $r \geq 1$ and all $p \in \mathcal{P}(\mathbb{C}^{n+m})$,

$$(2.17) \quad \sup_{\mathbb{B}_{re}^n} |p_f| \leq c(\deg p) A(r)^{\deg p} \sup_{\mathbb{B}^n} |p_f| \leq C(r)^{\mu(\deg p)} \sup_{\mathbb{B}_r^n} |p_f|.$$

Inequality (2.17) and the Hadamard three circle theorem imply (see, e.g., [B, Sect. 3.1]), for each $0 < r < 1$ and all $p \in \mathcal{P}(\mathbb{C}^{n+m})$,

$$\sup_{\mathbb{B}_{re}^n} |p_f| \leq C(1)^{\mu(\deg p)} \sup_{\mathbb{B}_r^n} |p_f|.$$

Thus, Γ_f admits the Bernstein type inequality of the exponent μ .

(b) Suppose that condition (1.2) is valid for a compact nonpluripolar set $K \subset \mathbb{B}_{r_0}^n$ for some $r_0 > 0$. Then for each $r \geq r_0$ and all $p \in \mathcal{P}(\mathbb{C}^{n+m})$, $p|_{\Gamma_f} \not\equiv 0$,

$$\frac{\sup_{\mathbb{B}_{er}^n} |p_f|}{\sup_{\mathbb{B}_r^n} |p_f|} \leq \frac{\sup_{\mathbb{B}_{er}^n} |p_f|}{\sup_{\mathbb{B}_{r_0}^n} |p_f|} \leq \frac{\sup_{\mathbb{B}_{er}^n} |p_f|}{\sup_K |p_f|} \leq C(K; r)^{\mu(\deg p)}.$$

For $r < r_0$ a similar inequality with $C(K; r)$ replaced by $C(K; r_0)$ follows from that for $r = r_0$ by the Hadamard three circle theorem. Thus, Γ_f admits the Bernstein type inequality of exponent μ .

Conversely, assume that Γ_f admits the Bernstein type inequality of exponent μ . Suppose that $K \Subset \mathbb{B}_{r_0}^n$, $r_0 > 0$, is a compact nonpluripolar set. We set

$$t_K := \inf \left\{ \frac{1}{\mu(\deg p)} \ln \frac{\sup_K |p_f|}{\sup_{\mathbb{B}_{r_0}^n} |p_f|} \right\},$$

where the infimum is taken over all polynomials $p \in \mathcal{P}(\mathbb{C}^{n+m})$ such that $p|_{\Gamma_f} \not\equiv 0$.

Lemma 2.2. $t_K > -\infty$.

Proof. Assume, on the contrary, that $t_K = -\infty$. Then there exists a sequence of nonidentical zero on Γ_f polynomials $p_k \in \mathcal{P}(\mathbb{C}^{n+m})$ such that

$$\lim_{k \rightarrow \infty} \frac{1}{\mu(\deg p_k)} \ln \frac{\sup_K |p_{kf}|}{\sup_{\mathbb{B}_{r_0}^n} |p_{kf}|} = -\infty.$$

Let us consider the function

$$u(z) := \overline{\lim}_{k \rightarrow \infty} \frac{1}{\mu(\deg p_k)} \ln \frac{|p_{kf}(z)|}{\sup_{\mathbb{B}_{r_0}^n} |p_{kf}|}, \quad z \in \mathbb{C}^n.$$

Then the Bernstein type inequality (1.1) implies that for each $r \geq r_0$ there exists a real number $\tilde{c}(r)$ such that

$$\sup_{\mathbb{B}_r^n} u \leq \tilde{c}(r).$$

Let u^* be the upper semicontinuous regularization of u . The previous inequality and the Hartogs lemma on subharmonic functions imply that u^* is a nonidentical $-\infty$ plurisubharmonic function on \mathbb{C}^n such that $\sup_{\mathbb{B}_{r_0}^n} u^* = 0$. Moreover, $u|_K = -\infty$ and the set $S \subset \mathbb{C}^n$ where u differs from u^* is pluripolar, see [BT, Th. 4.2.5]. Since by the hypothesis K is nonpluripolar, $K \setminus S$ is nonpluripolar as well. Thus $u^* = -\infty$ on the nonpluripolar set $K \setminus S$ and so it equals $-\infty$ everywhere, a contradiction proving the lemma. \square

Lemma 2.2 and the Bernstein type inequality show that for all $p \in \mathcal{P}(\mathbb{C}^{n+m})$, $r \geq r_0$,

$$\begin{aligned} \sup_{\mathbb{B}_r^n} |p_f| &\leq C(re^{-1})^{\mu(\deg p)} \sup_{\mathbb{B}_{re^{-1}}^n} |p_f| \leq \cdots \leq \left(\prod_{i=1}^{\lfloor \ln \frac{r}{r_0} \rfloor + 1} C(re^{-i})^{\mu(\deg p)} \right) \sup_{\mathbb{B}_{r_0}^n} |p_f| \\ &\leq C(K; r)^{\mu(\deg p)} \sup_K |p_f|, \end{aligned}$$

where

$$C(K; r) := e^{-t_K} \left(\prod_{i=1}^{\lfloor \ln \frac{r}{r_0} \rfloor + 1} C(re^{-i}) \right).$$

Also, for $0 < r < r_0$ we obviously have

$$\sup_{\mathbb{B}_r^n} |p_f| \leq \sup_{\mathbb{B}^n} |p_f| \leq (e^{-t_K})^{\mu(\deg p)} \sup_K |p_f|.$$

This completes the proof of (b).

(c) By $\mathcal{P}_k(\mathbb{C}^N) \subset \mathcal{P}(\mathbb{C}^N)$ we denote the space of holomorphic polynomials of degree at most k . Then

$$\dim_{\mathbb{C}}(\mathcal{P}_k(\mathbb{C}^N)) = \binom{N+k}{N} =: d_{k,N}.$$

Assume without loss of generality that the coordinate $f_1 : \mathbb{C}^n \rightarrow \mathbb{C}$ of the holomorphic map $f = (f_1, \dots, f_n) : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is nonpolynomial (for otherwise, if all f_i are polynomials, then Γ_f is algebraic).

In what follows for an entire function g on \mathbb{C}^n by $\sum_{|\alpha|=0}^{\infty} [g]_{\alpha} z^{\alpha}$ we denote its Taylor series at $0 \in \mathbb{C}^n$. Here $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $z^{\alpha} := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$.

We set

$$(2.18) \quad s_k := \left\lfloor \frac{k^{1+\frac{1}{n}}}{(n+2)^{\frac{1}{n}}} \right\rfloor.$$

Since

$$\lim_{k \rightarrow \infty} \frac{d_{s_k, n}}{k^{n+1}} = \frac{1}{(n+2)n!} < \frac{1}{(n+1)!} = \lim_{k \rightarrow \infty} \frac{d_{k, n+1}}{k^{n+1}},$$

there is $k_0 \in \mathbb{N}$ such that $d_{s_k, n} < d_{k, n+1}$ for all $k \geq k_0$.

Lemma 2.3. *For every $k \geq k_0$ there exists a polynomial $P_k \in \mathcal{P}_k(\mathbb{C}^{n+m})$ such that $P_{kf} := P_k(\cdot, f(\cdot)) \not\equiv 0$ whose Taylor series at $0 \in \mathbb{C}^n$ has a form*

$$P_{kf}(z) = \sum_{|\alpha|=s_k+1}^{\infty} [P_{kf}]_{\alpha} z^{\alpha}, \quad z \in \mathbb{C}^n.$$

Proof. Since $d_{s_k, n} < d_{k, n+1}$, the linear map $\pi : \mathcal{P}_k(\mathbb{C}^{n+1}) \rightarrow \mathcal{P}_{s_k}(\mathbb{C}^n)$,

$$\pi(p)(z) := \sum_{|\alpha|=0}^{s_k} [p_{f_1}]_{\alpha} z^{\alpha}, \quad p_{f_1} := p(\cdot, f_1(\cdot)), \quad z \in \mathbb{C}^n,$$

has a nonzero kernel. Then as P_k we choose the pullback of a nonzero element of $\ker \pi$ to \mathbb{C}^{n+m} with respect to the natural projection $\mathbb{C}^{n+m} \rightarrow \mathbb{C}^{n+1}$ onto the first $n+1$ coordinates. Since f_1 is nonpolynomial, $P_{kf} \neq 0$. \square

(!) In what follows by \mathcal{L}_n we denote the family of complex lines $l \subset \mathbb{C}^n$ passing through the origin.

For $r > 0$, let $l_r \in \mathcal{L}_n$ be a complex line such that

$$\sup_{l_r \cap \mathbb{B}_r^n} |P_{kf}| = M_{P_{kf}}(r).$$

Let us identify l_r with \mathbb{C} . Then the univariate entire function $h_k := P_{kf}|_{l_r}$ has zero of order at least $s_k + 1$ at 0. Let $n_{h_k}(r)$ denote the number of zeros of h_k in $l_r \cap \mathbb{B}_r^n = \mathbb{D}_r$ counted with their multiplicities. Then due to the Jensen type inequality, see [VP, Lm. 1], and the Bernstein type inequality of exponent μ for Γ_f (cf. (1.1)),

$$s_k + 1 \leq n_{h_k}(r) \leq \frac{m_{h_k}(er) - m_{h_k}(r)}{\ln\left(\frac{1+\epsilon^2}{2e}\right)} \leq \frac{5}{2} \ln\left(\frac{M_{P_{kf}}(er)}{M_{P_{kf}}(r)}\right) \leq \frac{5}{2} \mu(k) \ln C(r).$$

Choosing here $r = 1$ we obtain, cf. (2.18), that there exists $c > 0$ such that for all $k \geq k_0$

$$\mu(k) \geq ck^{1+\frac{1}{n}}.$$

This implies the required statement:

$$\lim_{k \rightarrow \infty} \frac{\mu(k)}{k^{1+\frac{1}{n}}} \neq 0.$$

(d) Let $C_i(r)$, $r > 0$, be the constant in the Bernstein type inequality of exponent μ_i for Γ_f , $i = 1, 2$. Then for all $p \in \mathcal{P}_k(\mathbb{C}^{n+m}) \setminus \{0\}$, $r > 0$ and all $\mu \geq \min\{\mu_1, \mu_2\}$ we have

$$\begin{aligned} \frac{\sup_{\mathbb{B}_{er}^n} |pf|}{\sup_{\mathbb{B}_r^n} |pf|} &\leq \min_{i=1,2} \left\{ C_i(r)^{\mu_i(\deg p)} \right\} \leq \min_{i=1,2} \left\{ (\max(C_1(r), C_2(r)))^{\mu_i(\deg p)} \right\} \\ &= (\max(C_1(r), C_2(r)))^{\min_{i=1,2} \{\mu_i(\deg p)\}} \leq (\max(C_1(r), C_2(r)))^{\mu(\deg p)}. \end{aligned}$$

This gives the required statement.

(e) For $w \neq 0$ we set $d_w := \|w\|$. Then for each $r \geq d_w + 1$ the open ball $\mathbb{B}_r^n(w)$ contains the ball $\mathbb{B}_{r-d_w}^n$ and the open ball $\mathbb{B}_{er}^n(w)$ is contained in the ball $\mathbb{B}_{er+d_w}^n$. Hence, for all

$p \in \mathcal{P}_k(\mathbb{C}^{n+m}) \setminus \{0\}$ and all $r \geq d_w + 1$ we obtain, for $s := \left\lfloor \ln \left(\frac{er+d_w}{r-d_w} \right) \right\rfloor + 1$,

$$(2.19) \quad \frac{\sup_{\mathbb{B}_{er}^n} |p_{f_w}|}{\sup_{\mathbb{B}_r^n} |p_{f_w}|} := \frac{\sup_{\mathbb{B}_{er}^n(w)} |p_f|}{\sup_{\mathbb{B}_r^n(w)} |p_f|} \leq \frac{\sup_{\mathbb{B}_{er+d_w}^n} |p_f|}{\sup_{\mathbb{B}_{r-d_w}^n} |p_f|} \\ \leq \left(\prod_{j=1}^{s-1} C(e^j(r-d_w)) \right)^{\mu(\deg p)} =: C(r, w)^{\mu(\deg p)}.$$

From here, using the Hadamard three circle theorem, for all $r \in (0, d_w + 1)$ we get

$$(2.20) \quad \frac{\sup_{\mathbb{B}_{er}^n} |p_{f_w}|}{\sup_{\mathbb{B}_r^n} |p_{f_w}|} := \frac{\sup_{\mathbb{B}_{er}^n(w)} |p_f|}{\sup_{\mathbb{B}_r^n(w)} |p_f|} \leq C(d_w + 1, w)^{\mu(\deg(p))}.$$

Inequalities (2.19) and (2.20) show that f_w admits the Bernstein type inequality of exponent μ as well.

(f) By definition,

$$\Gamma_{f_1 \times f_2} = \{(z, f_1(z), w, f_2(w)) : z \in \mathbb{C}^{n_1}, w \in \mathbb{C}^{n_2}\} \subset \mathbb{C}^{n_1+n_2+m_1+m_2}.$$

For each $r > 0$ and $p \in \mathcal{P}(\mathbb{C}^{n_1+n_2+m_1+m_2})$ applying Bernstein type inequalities of exponents μ_1 and μ_2 to restrictions of p to cross sections $\Gamma_{f_1} \times \{(w, f_2(w))\}$ and $\{(z, f_1(z))\} \times \Gamma_{f_2}$, for fixed $z \in \mathbb{C}^{n_1}$, $w \in \mathbb{C}^{n_2}$, we get

$$\begin{aligned} \sup_{\mathbb{B}_{er}^{n_1} \times \mathbb{B}_{er}^{n_2}} |p_{f_1 \times f_2}| &\leq C_1(r)^{\mu_1(\deg p)} \sup_{\mathbb{B}_r^{n_1} \times \mathbb{B}_{er}^{n_2}} |p_{f_1 \times f_2}| \\ &\leq C_1(r)^{\mu_1(\deg p)} C_2(r)^{\mu_2(\deg p)} \sup_{\mathbb{B}_r^{n_1} \times \mathbb{B}_r^{n_2}} |p_{f_1 \times f_2}| \\ &\leq (C_1(r) C_2(r))^{\max_{i=1,2} \{\mu_i(\deg p)\}} \sup_{\mathbb{B}_r^{n_1} \times \mathbb{B}_r^{n_2}} |p_{f_1 \times f_2}|. \end{aligned}$$

Replacing products of balls by suitable inscribed and circumscribed balls of $\mathbb{C}^{n_1+n_2+m_1+m_2}$ and arguing as in the proof of (e) we obtain that $\Gamma_{f_1 \times f_2}$ admits the Bernstein type inequality of exponent $\max(\mu_1, \mu_2)$.

Now, assume that $\Gamma_{f_1 \times f_2}$ admits the Bernstein type inequality of exponent μ . Applying this inequality to polynomials p pulled back from $\mathbb{C}^{n_i+m_i}$ by means of the natural projections $\mathbb{C}^{n_1+n_2+m_1+m_2} \rightarrow \mathbb{C}^{n_i+m_i}$, $i = 1, 2$, we obtain

$$\sup_{\mathbb{B}_{er}^{n_i}} |p_{f_i}| = \sup_{\mathbb{B}_{er}^{n_1+n_2}} |p_{f_1 \times f_2}| \leq C(r)^{\mu(\deg p)} \sup_{\mathbb{B}_r^{n_1+n_2}} |p_{f_1 \times f_2}| = C(r)^{\mu(\deg p)} \sup_{\mathbb{B}_r^{n_i}} |p_{f_i}|.$$

Thus, Γ_{f_i} , $i = 1, 2$, admit the Bernstein type inequality of exponent μ .

The proof of the theorem is complete.

3. PROOF OF THEOREM 1.2

Proof. (a) Approximating polynomial $p = 1$ by the sequence of polynomials of degree k $\{p_i\}_{i \in \mathbb{N}}$,

$$(3.21) \quad p_i(z) := \frac{i + z_1^k}{\sup_{z \in K} (i + z_1^k)}, \quad z = (z_1, \dots, z_{n+m}) \in \mathbb{C}^{n+m},$$

we conclude that $u_{K,\mu}^k \geq 0$. Then for $z_1, z_2 \in \mathbb{C}^n$ we have

$$\begin{aligned} & u_{K,\mu}^k(z_1; f) - u_{K,\mu}^k(z_2; f) \\ & \leq \sup \left\{ \frac{\ln^+ |p_f(z_1)| - \ln^+ |p_f(z_2)|}{\max(1, \mu(k))} : p \in \mathcal{P}(\mathbb{C}^{n+m}), \deg p = k, \sup_K |p_f| = 1 \right\} \\ & \leq \sup \left\{ \frac{|p_f(z_1) - p_f(z_2)|}{\max(1, \mu(k))} : p \in \mathcal{P}(\mathbb{C}^{n+m}), \deg p = k, \sup_K |p_f| = 1 \right\} \\ & \leq \sup \left\{ \frac{C(z_1, z_2) \|z_1 - z_2\|}{\max(1, \mu(k))} : p \in \mathcal{P}(\mathbb{C}^{n+m}), \deg p = k, \sup_K |p_f| = 1 \right\} \\ & \leq C(z_1, z_2) \|z_1 - z_2\|. \end{aligned}$$

Here we use that $\ln^+ x := \max(0, \ln x)$, $x > 0$, is a Lipschitz function with Lipschitz constant 1 and the uniform boundedness of the family

$$\{p_f : p \in \mathcal{P}(\mathbb{C}^{n+m}), \deg p = k, \sup_K |p_f| = 1\}|_U \subset C(U)$$

on each compact subset $U \subset \mathbb{C}^n$. The constant C in the above inequality is obtained by applying the Cauchy estimates for derivatives of p_f on an open polydisk containing z_1 and z_2 .

The above inequality shows that the function $u_{K,\mu}^k$ is locally Lipschitz and, in particular, it is continuous. Then, by definition, it is plurisubharmonic.

(b) Clearly, if Γ_f admits the Bernstein type inequality of exponent μ , then the function $u_{K,\mu}$ is locally bounded from above. Conversely, assume that the function $u_{K,\mu}$ is locally bounded from above. Then according to the Hartogs lemma on subharmonic functions, the sequence of continuous plurisubharmonic functions $\{u_{K,\mu}^k\}_{k \in \mathbb{N}}$ is uniformly bounded from above on each compact subset of \mathbb{C}^n . This implies fulfillment of inequality (1.2) and so due to Theorem 1.1 (b), Γ_f admits the Bernstein type inequality of exponent μ .

(c) Suppose that $u_{K,\mu,\bar{k}} \not\equiv 0$ for every subsequence $\bar{k} \subset \mathbb{N}$ but μ is not optimal. Then there exists a function $\mu_1 : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ such that $\mu_1 \leq c\mu$ for some $c > 0$,

$$\overline{\lim}_{k \rightarrow \infty} \frac{\mu(k)}{\mu_1(k)} = \infty$$

and Γ_f admits the Bernstein type inequality of exponent μ_1 . Let $\bar{k} = \{k_i\}_{i \in \mathbb{N}} \subset \mathbb{N}$ be a subsequence such that

$$\lim_{i \rightarrow \infty} \frac{\mu(k_i)}{\mu_1(k_i)} = \infty.$$

We have, cf. Theorem 1.1 (b),

$$\begin{aligned} 0 \leq u_{K,\mu;\bar{k}}(z; f) &:= \overline{\lim}_{i \rightarrow \infty} \sup \left\{ \frac{\ln |p_f(z)|}{\max(1, \mu(k_i))} : p \in \mathcal{P}(\mathbb{C}^{n+m}), \deg p = k_i, \sup_K |p_f| = 1 \right\} \\ &= \overline{\lim}_{i \rightarrow \infty} \sup \left\{ \frac{\max(1, \mu_1(k_i))}{\max(1, \mu(k_i))} \cdot \frac{\ln |p_f(z)|}{\max(1, \mu_1(k_i))} : p \in \mathcal{P}(\mathbb{C}^{n+m}), \deg p = k_i, \sup_K |p_f| = 1 \right\} \\ &\leq \overline{\lim}_{i \rightarrow \infty} \frac{\max(1, \mu_1(k_i))}{\max(1, \mu(k_i))} \ln C(K; \|z\|) = 0. \end{aligned}$$

Here $C(K; r)$, $r > 0$, is the constant in (1.2) for the exponent μ_1 .

This implies that $u_{K,\mu;\bar{k}} = 0$, a contradiction showing that μ is optimal.

Conversely, suppose that Γ_f admits the Bernstein type inequality of an optimal exponent μ but there exists a subsequence $\bar{k} = \{k_i\}_{i \in \mathbb{N}} \subset \mathbb{N}$ such that $u_{K,\mu;\bar{k}} = 0$. Then the Hartogs lemma on subharmonic functions implies that for each $\ell \in \mathbb{N}$ there exists a number $i(\ell) \in \mathbb{N}$ such that for all $i \geq i(\ell)$

$$(3.22) \quad \sup_{\mathbb{B}_\ell^n} u_{K,\mu}^{k_i} \leq \frac{1}{\ell}.$$

Passing to a subsequence, if necessary, we may assume that $\{i(\ell)\}_{\ell \in \mathbb{N}}$ is an increasing sequence. We set $\bar{k}_* := \{k_{i(\ell)}\}_{\ell \in \mathbb{N}}$. Let us define a function $\mu_1 : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ by the formula

$$\mu_1(k) = \begin{cases} \mu(k) & \text{if } k \notin \bar{k}_* \\ \frac{\mu(k)}{\ell} & \text{if } k = k_{i(\ell)} \text{ for some } \ell \in \mathbb{N}. \end{cases}$$

Then due to (3.22) we obtain

$$\begin{aligned} u_{K,\mu_1}(z; f) &= \overline{\lim}_{k \rightarrow \infty} u_{K,\mu_1}^k(z; f) = \max \left(\overline{\lim}_{\substack{k \rightarrow \infty \\ k \notin \bar{k}_*}} u_{K,\mu}^k(z; f), \overline{\lim}_{\ell \rightarrow \infty} \ell u_{K,\mu}^{i(\ell)}(z; f) \right) \\ &\leq \max(u_{K,\mu}(z; f), 1). \end{aligned}$$

Thus, u_{K,μ_1} is locally bounded from above and so part (b) of the theorem implies that Γ_f admits the Bernstein type inequality of exponent μ_1 . Clearly, $\mu_1 \leq \mu$. Thus, due to the optimality of μ , function μ_1 must be equivalent to μ . However, this is wrong as $\overline{\lim}_{k \rightarrow \infty} \frac{\mu(k)}{\mu_1(k)} = \infty$. This contradiction shows that $u_{K,\mu;\bar{k}} \not\equiv 0$ for every subsequence $\bar{k} \subset \mathbb{N}$.

(d) If Γ_f admits the Bernstein type inequality of exponent μ , then due to Theorem 1.1 (b), $u_K^k(r; f) \leq \ln C(K; r) \mu(k)$ for all $k \in \mathbb{Z}_+$, $r > 0$. This implies that $\langle u_K(r; f) \rangle \leq \langle \mu \rangle$ for all $r > 0$. Assume, in addition, that $\langle \mu \rangle \in \mathcal{U}_f^K$. Then $\langle \mu \rangle$ is the maximal element of

\mathcal{U}_f^K . If μ is not optimal for Γ_f , then there is an exponent μ_1 for Γ_f such that $\langle \mu_1 \rangle < \langle \mu \rangle$. Since $\langle \mu \rangle \in \mathcal{U}_f^K$, we must have $\langle \mu \rangle \leq \langle \mu_1 \rangle$, a contradiction showing that μ is optimal.

Conversely, suppose that μ is an optimal exponent for Γ_f . We require

Lemma 3.1. *There exist $r > 0$ and $c > 0$ such that for all $k \in \mathbb{N}$*

$$u_K^k(r; f) \geq c \max(1, \mu(k)).$$

Proof. Assume, on the contrary, that for each $r > 0$ and $c > 0$ there exists an integer $k_{r,c} \in \mathbb{N}$ such that

$$(3.23) \quad u_K^{k_{r,c}}(r; f) < c \max(1, \mu(k_{r,c})).$$

Choose $r := j$, $c := \frac{1}{j}$ and set $k_j := k_{j,1/j}$, $j \in \mathbb{N}$. Let us show that $\overline{\lim}_{j \rightarrow \infty} k_j = \infty$. Indeed, for otherwise, the sequence $\bar{k} := \{k_j\}_{j \in \mathbb{N}}$ is bounded. In particular, there exists an element $k' \geq 1$ of \bar{k} such that (as each $u_{K,\mu}^k$ is plurisubharmonic) $u_{K,\mu}^{k'} = 0$, a contradiction.

Since each $u_{K,\mu}^k$ is plurisubharmonic, inequality (3.23) implies that the function

$$u_{K,\mu;\bar{k}} := \overline{\lim}_{j \rightarrow \infty} u_{K,\mu}^{k_j}$$

is identically zero. Due to part (c) of the theorem, this contradicts the optimality of μ .

The proof of the lemma is complete. \square

As the corollary of the lemma we get

$$(\langle \mu \rangle \geq) \langle u_K(r; f) \rangle \geq \langle \mu \rangle.$$

Thus $\langle \mu \rangle = \langle u_K(r; f) \rangle \in \mathcal{U}_f^K$ is maximal.

The proof of the theorem is complete. \square

4. PROOF OF THEOREM 1.4

In the proof of the theorem we use the following Bernstein type inequality for exponential polynomials established in [VP].

Let

$$g(z) = \sum_{j=1}^n p_j(z) e^{q_j z}, \quad z \in \mathbb{C},$$

where $p_j \in \mathcal{P}(\mathbb{C})$, $\deg p_j = d_j$ and $q_j \in \mathbb{C}$ are pairwise disjoint, $1 \leq j \leq n$, be an exponential polynomial on \mathbb{C} . The expression

$$m(g) := \sum_{j=1}^n (1 + d_j)$$

is called the *degree* of g . In turn, the exponential type of g is defined by the formula

$$\epsilon(g) := \max_{1 \leq j \leq n} |q_j|.$$

Then [VP, p. 27, Eq. (21)] asserts that for each $r > 0$,

$$(4.24) \quad \sup_{\mathbb{D}_{er}} |g| \leq e^{m(g)+2er\epsilon(g)} \sup_{\mathbb{D}_r} |g|.$$

Proof of Theorem 1.4. First, we prove the theorem for $l = 1$ and P, Q the identity automorphisms. In this case,

$$f_{P,Q}(z) = f(z) = (e^{\alpha_1 z}, \dots, e^{\alpha_m z}), \quad z \in \mathbb{C},$$

where $\alpha_1, \dots, \alpha_m$ are linearly independent over \mathbb{Q} complex numbers. Let $p \in \mathcal{P}_k(\mathbb{C}^{m+1})$,

$$p(z, w) := \sum_{|\gamma| \leq k} c_\gamma z^{\gamma_1} w_1^{\gamma_2} \dots w_m^{\gamma_{m+1}}, \quad z \in \mathbb{C}, \quad w = (w_1, \dots, w_m) \in \mathbb{C}^m$$

(here $\gamma = (\gamma_1, \dots, \gamma_{m+1}) \in \mathbb{Z}_+^{m+1}$ and all $c_\gamma \in \mathbb{C}$).

Since $\alpha_1, \dots, \alpha_m$ are linearly independent over \mathbb{Q} ,

$$p_f(z) := p(z, f(z)) = \sum_{|\gamma| \leq k} c_\gamma z^{\gamma_1} e^{(\gamma_2 \alpha_1 + \dots + \gamma_{m+1} \alpha_m)z} = \sum_{|\gamma'| \leq k} p_{\gamma'}(z) e^{(\gamma_2 \alpha_1 + \dots + \gamma_{m+1} \alpha_m)z},$$

here $\gamma' := (\gamma_2, \dots, \gamma_{m+1})$ and $p_{\gamma'} \in \mathcal{P}_{k-|\gamma'|}(\mathbb{C})$.

Then the exponential type of p_f is

$$\epsilon(p_f) := \max_{|\gamma| \leq k} \left| \sum_{j=2}^{m+1} \gamma_j \alpha_{j-1} \right| \leq k \max_{1 \leq j \leq m} |\alpha_j| =: k\bar{\alpha}$$

and the degree of p_f satisfies the inequality

$$m(p_f) \leq \sum_{|\gamma'| \leq k} (1 + (k - |\gamma'|)) = \sum_{|\gamma'| \leq k} 1 = d_{k,m+1} =: \binom{m+1+k}{k}.$$

Hence, in this case (4.24) yields the inequality

$$(4.25) \quad \sup_{\mathbb{D}_{er}} |p_f| \leq e^{2erk\bar{\alpha} + d_{k,m+1}} \sup_{\mathbb{D}_r} |p_f|, \quad r > 0.$$

Note that for all $k \geq 1$

$$(e^{2erk\bar{\alpha} + d_{k,m+1}})^{\frac{1}{k^{m+1}}} < 5^{m+1} e^{2er\bar{\alpha}} =: C(r, f).$$

This and (4.25) show that Γ_f satisfies the Bernstein type inequality of exponent μ_{id}^{m+1} .

Next, we show that this exponent is optimal.

Since complex numbers $\alpha_1, \dots, \alpha_m$ are linearly independent over \mathbb{Q} , the restriction maps $\mathcal{P}_k(\mathbb{C}^{m+1}) \rightarrow \mathcal{P}_k(\mathbb{C}^{m+1})|_{\Gamma_f}$, $k \in \mathbb{Z}_+$, are linear isomorphisms. Thus arguing as in the proof of Theorem 1.1 (c) we conclude that for each $k \geq 1$ there exists a polynomial $g_k \in \mathcal{P}_k(\mathbb{C}^{m+1})$ such that $g_{kf} \neq 0$ and has zero of multiplicity $d_{k,m+1} - 1$ at $0 \in \mathbb{C}$. Then due to the Jensen

type inequality, see [VP, Lm. 1], and the Bernstein type inequality of exponent μ for Γ_f (cf. the proof of Theorem 1.1 (c) for similar arguments),

$$d_{k,m+1} - 1 \leq \frac{m_{g_k}(er) - m_{g_k}(r)}{\ln\left(\frac{1+e^2}{2e}\right)} \leq \frac{5}{2}\mu(k) \ln C(r).$$

Taking here $r = 1$ we obtain that there exists $c > 0$ such that for all $k \geq 1$

$$\mu(k) \geq ck^{m+1} := c\mu_{\text{id}}^{m+1}(k).$$

This and (4.25) show that μ_{id}^{m+1} is the optimal exponent for the Bernstein type inequality on Γ_f completing the proof of the theorem in this particular case.

We deduce the general case from the one just proved by means of the following result.

Lemma 4.1. *Suppose $\Gamma_f \subset \mathbb{C}^{n+m}$, $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$, admits the Bernstein type inequality of exponent μ_{id}^r , $r \geq 1$. Let P and Q be holomorphic polynomial automorphisms of \mathbb{C}^n and \mathbb{C}^m , respectively. Then $f_{P,Q} := Q \circ f \circ P$ admits the Bernstein type inequality of exponent μ_{id}^r as well. Moreover, if μ_{id}^r is optimal for Γ_f , then it is optimal for $\Gamma_{f_{P,Q}}$ as well.*

Proof. By definition, there are some $s, t \in \mathbb{N}$ such that the coordinates of maps $P^{\pm 1}$ and $Q^{\pm 1}$ are holomorphic polynomials in $\mathcal{P}_s(\mathbb{C}^n)$ and $\mathcal{P}_t(\mathbb{C}^m)$, respectively. Then the correspondence $h(z, w) \mapsto h(P^{-1}(z), Q(w))$, $z \in \mathbb{C}^n$, $w \in \mathbb{C}^m$, determines a linear injective map $I : \mathcal{P}_k(\mathbb{C}^{n+m}) \rightarrow \mathcal{P}_{k \max(s,t)}(\mathbb{C}^{n+m})$. By K we denote the image of the closure of \mathbb{B}^n under map P , i.e. $K := P(\text{cl}(\mathbb{B}^n))$. Since P is a holomorphic automorphism of \mathbb{C}^n , K is a nonpluripolar compact subset of \mathbb{C}^n . By definition, cf. Theorem 1.2,

$$\begin{aligned} u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r}^k(z; f_{P,Q}) &= \sup \left\{ \frac{\ln |h_{f_{P,Q}}(z)|}{\max(1, k^r)} : h \in \mathcal{P}(\mathbb{C}^{n+m}), \deg h = k, \sup_{\mathbb{B}^n} |h_{f_{P,Q}}| = 1 \right\} \\ &= \sup \left\{ \frac{\ln |(I(h))_f(P(z))|}{\max(1, k^r)} : h \in \mathcal{P}(\mathbb{C}^{n+m}), \deg h = k, \sup_K |(I(h))_f| = 1 \right\} \\ &\leq \sup \left\{ \frac{\ln |g_f(P(z))|}{\max(1, k^r)} : g \in \mathcal{P}(\mathbb{C}^{n+m}), \deg g = k \max(s, t), \sup_K |g_f| = 1 \right\} \\ &\leq (\max(s, t))^r u_{K, \mu_{\text{id}}^r}^{k \max(s,t)}(P(z); f), \quad z \in \mathbb{C}^n. \end{aligned}$$

This yields

$$\begin{aligned} (4.26) \quad u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r}^k(z; f_{P,Q}) &:= \overline{\lim}_{k \rightarrow \infty} u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r}^k(z; f_{P,Q}) \leq (\max(s, t))^r \overline{\lim}_{k \rightarrow \infty} u_{K, \mu_{\text{id}}^r}^{k \max(s,t)}(P(z); f) \\ &\leq (\max(s, t))^r u_{K, \mu_{\text{id}}^r}(P(z); f), \quad z \in \mathbb{C}^n. \end{aligned}$$

Since the function $u_{K, \mu_{\text{id}}^r}(\cdot; f)$ is locally bounded from above by Theorem 1.2 (b), the latter inequality implies that the function $u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r}(\cdot; f_{P,Q})$ is locally bounded from above

as well. Then by Theorem 1.2 (b) graph $\Gamma_{f_{P,Q}}$ admits the Bernstein type inequality of exponent μ_{id}^r .

Further, suppose that μ_{id}^r is optimal for Γ_f . Let $\bar{k} = \{k_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ be a subsequence. Applying the arguments similar to the above one to functions $f_{P,Q} = Q \circ f \circ P$ and automorphisms P^{-1}, Q^{-1} instead of f and P, Q as in the hypothesis of the lemma, we get for $\bar{k}_{s,t} := \{k_j \max(s, t)\}_{j \in \mathbb{N}}$ (cf. (4.26))

$$(4.27) \quad \begin{aligned} u_{K, \mu_{\text{id}}^r; \bar{k}}(z; f) &:= \overline{\lim}_{j \rightarrow \infty} u_{K, \mu_{\text{id}}^r}^{k_j}(z; f) \leq (\max(s, t))^r \overline{\lim}_{j \rightarrow \infty} u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r}^{k_j \max(s, t)}(P^{-1}(z); f_{P,Q}) \\ &= (\max(s, t))^r u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r; \bar{k}_{s,t}}(P^{-1}(z); f_{P,Q}), \quad z \in \mathbb{C}^n. \end{aligned}$$

Since μ_{id}^r is optimal for Γ_f , Theorem 1.2 (c) implies that $u_{K, \mu_{\text{id}}^r; \bar{k}}(\cdot; f) \not\equiv 0$. Hence, equation (4.27) shows that $u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r; \bar{k}_{s,t}}(\cdot; f_{P,Q}) \not\equiv 0$ as well (recall that all functions in (4.27) are nonnegative).

Let us check a similar statement for an arbitrary sequence $\bar{n} = \{n_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$.

We set

$$\bar{k} = \{k_j\}_{j \in \mathbb{N}}, \quad k_j := \left\lfloor \frac{n_j}{\max(s, t)} \right\rfloor, \quad j \in \mathbb{N}.$$

Then, by the definition of $u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r}^k(\cdot; f_{P,Q})$,

$$u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r}^{k_j \max(s, t)}(z; f_{P,Q}) \leq \frac{\max(1, n_j^r)}{\max(1, (k_j \max(s, t))^r)} u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r}^{n_j}(z; f_{P,Q}) \leq 2^r u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r}^{n_j}(z; f_{P,Q}).$$

This yields

$$(4.28) \quad u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r; \bar{k}_{s,t}}(z; f_{P,Q}) \leq 2^r u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r; \bar{n}}(z; f_{P,Q}).$$

Since $u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r; \bar{k}_{s,t}}(\cdot; f_{P,Q}) \not\equiv 0$, the latter shows that $u_{\text{cl}(\mathbb{B}^n), \mu_{\text{id}}^r; \bar{n}}(\cdot; f_{P,Q}) \not\equiv 0$ as well. Thus, Theorem 1.2 (c) implies that the exponent μ_{id}^r is optimal for $\Gamma_{f_{P,Q}}$.

The proof of the lemma is complete. \square

Now, if $f_j : \mathbb{C} \rightarrow \mathbb{C}^{m_j}$, $1 \leq j \leq l$, are exponential maps of maximal transcendence degrees, then due to Theorem 1.1 (f) and the above considered case of $l = 1$ and P, Q the identity automorphisms, the graph of $f_1 \times \cdots \times f_l : \mathbb{C}^l \rightarrow \mathbb{C}^{m_1 + \cdots + m_l}$ satisfies the Bernstein type inequality of exponent $\mu_{\text{id}}^{\bar{m}+1}$, $\bar{m} = \max_{1 \leq j \leq l} m_j$. Hence, by Lemma 4.1 graph $\Gamma_{F_{P,Q}}$, $F_{P,Q} := Q \circ (f_1 \times \cdots \times f_l) \circ P$, satisfies the Bernstein type inequality of exponent $\mu_{\text{id}}^{\bar{m}+1}$ as well. Since, by Theorem 1.1 (f), $\mu_{\text{id}}^{\bar{m}+1}$ is optimal for $f_1 \times \cdots \times f_l$, Lemma 4.1 implies that $\mu_{\text{id}}^{\bar{m}+1}$ is optimal for $\Gamma_{F_{P,Q}}$.

This completes the proof of the theorem. \square

5. PROOF OF THEOREM 1.5

According to [B, Th. 2.5 (c)] there exist increasing sequences $\{n_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$, $\{r_j\}_{j \in \mathbb{N}} \subset \mathbb{R}_+$ converging to ∞ and a nonincreasing sequence $\{\epsilon_j\} \subset \mathbb{R}_+$ converging to 0 such that for all

$g \in \mathcal{P}_{n_j}(\mathbb{C}^{n+1})$ and all $0 \leq r \leq r_j$,

$$(5.29) \quad M_{g_f}(er) \leq e^{C_{\rho_f} n_j^{2+\epsilon_j}} M_{g_f}(r),$$

for some constant $C_{\rho_f} > 0$ depending on the order of f only.

Without loss of generality we assume that $r_1 = 0$ and $n_1 = 1$

For $j \in \mathbb{N}$, $r \in [r_j, r_{j+1})$ we set

$$C_j(r) := \sup \left\{ \frac{M_{g_f}(er)}{M_{g_f}(r)} : g \in \mathcal{P}_{n_j}(\mathbb{C}^{n+1}) \setminus \{0\} \right\}.$$

Since f is nonpolynomial and the space $\mathcal{P}_{n_{j-1}}(\mathbb{C}^{n+1})$ is finite dimensional, each $C_j(r) < \infty$. We define

$$(5.30) \quad C(r) := \max(C_j(r), e^{C_{\rho_f}}) \quad \text{for } r_j \leq r < r_{j+1}, \quad j \in \mathbb{N}.$$

Lemma 5.1. *For all $g \in \mathcal{P}_{n_j}(\mathbb{C}^{n+1})$, $j \in \mathbb{N}$, and all $r > 0$*

$$(5.31) \quad M_{g_f}(er) \leq C(r) n_j^{2+\epsilon_j} M_{g_f}(r).$$

Proof. We consider two cases.

(1) If $0 \leq r < r_j$, then (5.31) follows from (5.29).

(2) If $r_k \leq r < r_{k+1}$ for some $k \geq j$, then

$$M_{g_f}(er) \leq C_k(r) M_{g_f}(r) \leq C(r) n_j^{2+\epsilon_j} M_{g_f}(r)$$

by the definition of $C(r)$.

The proof of the lemma is complete. \square

Now, let ν be an exponent in the Bernstein type inequality on Γ_f (existing by Theorem 1.1 (a)). We define a function $\mu : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ by the formula

$$\mu(k) = \begin{cases} \nu(k) & \text{if } k \notin \{n_j\}_{j \in \mathbb{N}} \\ n_j^{2+\epsilon_j} & \text{if } k = n_j, j \in \mathbb{N}. \end{cases}$$

Lemma 5.1 and the definition of the exponent ν easily imply that Γ_f satisfies the Bernstein type inequality of exponent μ . Note that

$$\liminf_{k \rightarrow \infty} \frac{\ln \mu(k)}{\ln k} \leq \liminf_{j \rightarrow \infty} \frac{(2 + \epsilon_j) \cdot \ln n_j}{\ln n_j} = 2.$$

This gives the right-hand side inequality of the theorem. The left-hand side inequality,

$$1 + \frac{1}{n} \leq \liminf_{k \rightarrow \infty} \frac{\ln \mu(k)}{\ln k},$$

follows from Theorem 1.1 (c).

The proof of the theorem is complete.

6. PROOFS OF THEOREMS 1.7 AND 1.8

6.1. Theorem A. In this part we discuss an auxiliary result used in the proofs of the theorems. For its formulation, we require some definitions.

In what follows, for each $l \in \mathcal{L}_n$, the family of complex lines passing through $0 \in \mathbb{C}^n$, we naturally identify $l \cap \mathbb{B}_r^n$ with \mathbb{D}_r .

Assume that $f : \mathbb{D}_r \rightarrow \mathbb{C}$ is holomorphic. The *Bernstein index* b_f of f is given by the formula

$$b_f(r) := \sup\{m_f(es) - m_f(s)\},$$

where the supremum is taken over all $\mathbb{D}_{es} \subseteq \mathbb{D}_r$. (We assume that $b_f(\cdot) = 0$ for $f = 0$.) The index is finite for all f defined in neighbourhoods of the closure of \mathbb{D}_r .

Let g be a holomorphic function in the domain $\mathbb{B}_{tr}^n \times \mathbb{D}_{3M} \subset \mathbb{C}^{n+1}$, $r \in \mathbb{R}_+$, $t \in [1, 9]$, $M \in \mathbb{R}_+ \cup \{\infty\}$ (here $\mathbb{D}_\infty := \mathbb{C}$). For every $l \in \mathcal{L}_n$ we determine

$$g_l := g|_{\Omega_l}, \quad \Omega_l := (l \cap \mathbb{B}_{tr}^n) \times \mathbb{D}_{3M}.$$

We write $g \in \mathcal{F}_{p,q}(r; t; M)$ for some $p, q \geq 0$ if

$$(6.32) \quad \begin{aligned} M_{g_l(\cdot, w)}(tr) &\leq e^p \cdot M_{g_l(\cdot, w)}(r) \quad \text{for all } l \in \mathcal{L}_n, w \in \mathbb{D}_{3M} \quad \text{and} \\ b_{g(z, \cdot)}(3M) &\leq q \quad \text{for all } z \in \mathbb{B}_{tr}^n. \end{aligned}$$

Example 6.1. One can easily check by means of the classical Bernstein inequality that a holomorphic polynomial of degree d on \mathbb{C}^{n+1} is in $\mathcal{F}_{p,q}(r; t; M)$ with $p = d \ln t$ and $q = d$. See [B] for other examples.

Theorem A (cf. [B, Theorem 2.8]). *Assume that f is of class \mathcal{C} . Then there exist numbers $k_0, r_0 \geq 1$, a continuous increasing to ∞ function $r : [k_0, \infty) \rightarrow [r_0, \infty)$ and a continuous decreasing to 0 function $\varepsilon : [k_0, \infty) \rightarrow \mathbb{R}_+$ such that for all $k \geq k_0$, $r(k) \geq r_0$, every $g \in \mathcal{F}_{p,q}(er(k); e; M_f(e^2r(k)))$ with $p \leq k$ and every $0 < r \leq r(k)$ the following inequalities hold for $g_f := g(\cdot, f(\cdot))$:*

(a)

$$\sup_{\mathbb{B}^n \times \mathbb{D}} |g| \leq e^{Ck^{1+\varepsilon(k)} \ln r(k) \max\{p,q\}} M_{g_f}(1);$$

(b)

$$\frac{M_{g_f}(er)}{M_{g_f}(r)} \leq e^{Ck^{1+\varepsilon(k)} \max\{p,q\}}.$$

Here for $\rho_f < \infty$ the constant C depends on the value of the limit superior of condition (1.7) and ρ_f , and for $\rho_f = \infty$ the constant $C = 1$.

Moreover,

(1) If $\rho_f < \infty$, then $\varepsilon = 0$ and function r is the right inverse of the nondecreasing function $k : [r_0, \infty) \rightarrow \mathbb{R}_+$,

$$(6.33) \quad k(r) := \frac{m_f(e^{-\alpha_{\rho_f} r}) - m_f(e^{-2\alpha_{\rho_f} r}) - 1}{9(\sqrt{e} + 1)^2(\rho_f^2 + 1)(17 + 2 \ln(\rho_f + 1))},$$

where $\alpha_{\rho_f} := \min(1, \ln(1 + \frac{1}{\rho_f}))$.

- (2) If $\rho = \infty$, then $r(k) = \frac{1}{e^2} m_f^{-1}(k^{1+\varepsilon''(k)}) \leq e^{k^{\delta(k)}}$, $k \geq k_0$, for some continuous functions $\varepsilon'', \delta : [k_0, \infty) \rightarrow \mathbb{R}_+$ decreasing to 0 as $k \rightarrow \infty$.

Proof. For $\rho_f < \infty$ the statement of the theorem is the direct consequence of Theorem 2.8 of [B]. The latter is proved under the assumption

$$(6.34) \quad \overline{\lim}_{t \rightarrow \infty} \left(\frac{\phi_f(t + \alpha_{\rho_f}) - \phi_f(t - \alpha_{\rho_f})}{\phi_f(t - \alpha_{\rho_f}) - \phi_f(t - 2\alpha_{\rho_f})} + \frac{\rho_f e^{\rho_f t}}{\phi_f(t - \alpha_{\rho_f}) - \phi_f(t - 2\alpha_{\rho_f})} \right) < A < \infty,$$

where the second summand is included only to give an effective upper bound of function r (see [B, Eq. (9.6)]). In particular, in this case the arguments of the proof of [B, Th. 2.8] imply that all statements of Theorem A are valid under the assumption

$$(6.35) \quad \overline{\lim}_{t \rightarrow \infty} \frac{\phi_f(t + \alpha_{\rho_f}) - \phi_f(t - \alpha_{\rho_f})}{\phi_f(t - \alpha_{\rho_f}) - \phi_f(t - 2\alpha_{\rho_f})} < A < \infty.$$

Note that since $\alpha_{\rho_f} \leq 1$, using that ϕ_f is a convex increasing function we obtain for $s := t - \alpha_{\rho_f}$,

$$(6.36) \quad \begin{aligned} & \frac{\phi_f(t + \alpha_{\rho_f}) - \phi_f(t - \alpha_{\rho_f})}{\phi_f(t - \alpha_{\rho_f}) - \phi_f(t - 2\alpha_{\rho_f})} \leq \frac{\frac{1}{2}(\phi_f(t+1) - \phi_f(t))}{\phi_f(s) - \phi_f(s-1)} \\ & = \frac{1}{2} \frac{\phi_f(t+1) - \phi_f(t)}{\phi_f(t) - \phi_f(t-1)} \cdot \frac{\phi_f(s+1) - \phi_f(s)}{\phi_f(s) - \phi_f(s-1)} \cdot \frac{\phi_f(t) - \phi_f(t-1)}{\phi_f(s+1) - \phi_f(s)} \\ & \leq \frac{1}{2} \frac{\phi_f(t+1) - \phi_f(t)}{\phi_f(t) - \phi_f(t-1)} \cdot \frac{\phi_f(s+1) - \phi_f(s)}{\phi_f(s) - \phi_f(s-1)}. \end{aligned}$$

Therefore if $f \in \mathcal{C}$ and satisfies (cf. (1.7))

$$\overline{\lim}_{t \rightarrow \infty} \frac{\phi_f(t+1) - \phi_f(t)}{\phi_f(t) - \phi_f(t-1)} < C,$$

then due to (6.36) inequality (6.35) is valid with $A := \frac{C^2}{2}$. This implication and [B, Th. 2.8] show that Theorem A is valid for $f \in \mathcal{C}$ with $\rho_f < \infty$.

Now, let us consider the case of $f \in \mathcal{C}$ with $\rho_f = \infty$. In this instance, the required result does not follow straightforwardly from Theorem 2.8 of [B] as it is proved under a weaker condition of $\psi = \phi_f$ in (1.8). To prove an analogous result (and therefore our Theorem A) in the general case, one follows the lines of the proof of Theorem 2.8. We just sketch the corresponding arguments leaving the details to the reader.

First, observe that condition (II) in the definition of class \mathcal{C} implies that there exists some $v_* \in \mathbb{R}$ and a continuous function $\kappa : [v_*, \infty) \rightarrow \mathbb{R}_+$, $\lim_{v \rightarrow \infty} \kappa(v) = 1$, such that

$$(6.37) \quad \ln \psi(v) = \kappa(v) \ln \phi_f(v), \quad v \geq v_*.$$

Now, for each sufficiently large $v \in \mathbb{R}$ by $s(v) \in \mathbb{R}_+$ we denote a number such that

$$(6.38) \quad \frac{\psi(v)}{\psi(v - 2s(v))} = e.$$

Since ψ is a continuous increasing function,

$$s(v) = \frac{1}{2} \left(v - \psi^{-1} \left(\frac{\psi(v)}{e} \right) \right),$$

i.e. s is a continuous in v function. Now, as in the proof of Theorem 2.8 (which uses only monotonicity and convexity of ϕ_f) we obtain for $\tilde{s}(v) := \min(s(v), 1)$

$$\frac{1}{s(v)} \leq \frac{2e\psi'(v)}{\psi(v)}$$

and, cf. (6.37),

$$(6.39) \quad \frac{1}{\tilde{s}(v)} \leq (\psi(v))^{\frac{\varepsilon(v)}{v}} = (m_f(e^v))^{\frac{\kappa(v)\varepsilon(v)}{v}}, \quad v \geq v_0,$$

where ε is a positive continuous function in v tending to 0 as $v \rightarrow \infty$ and $v_0 \in \mathbb{R}$ is sufficiently large.

Next, we determine continuous in v functions

$$t(v) := e^{\tilde{s}(v)}, \quad \tilde{r}(v) := e^{v-\tilde{s}(v)}.$$

Then as in the proof of Theorem 2.8 using properties of ψ we obtain (cf. [B, Eq. (8.36)]),

$$(6.40) \quad \begin{aligned} \ln \tilde{r}(v) &\leq \varepsilon'(v) [\ln \psi(v)]^2 \leq \varepsilon'(v) (\kappa(v))^2 [\ln \phi_f(v)]^2 \\ &= \varepsilon'(v) (\kappa(v))^2 [\ln(m_f(t(v)\tilde{r}(v)))]^2, \quad v \geq v_0, \end{aligned}$$

where ε' is a positive continuous function in v tending to 0 as $v \rightarrow \infty$.

Using (6.39) and (6.40) as in [B, Lm. 8.5] we have, for all sufficiently large v ,

$$(6.41) \quad k(t(v), \tilde{r}(v)) \leq \varepsilon''(v) (\ln m_f(t(v)\tilde{r}(v)))^2,$$

where ε'' is a positive continuous function in v tending to 0 as $v \rightarrow \infty$, and for such v

$$(6.42) \quad \frac{1}{\ln \left(\frac{1+t(v)}{2\sqrt{t(v)}} \right)} \leq 64 (m_f(t(v)\tilde{r}(v)))^{\frac{2\kappa(v)\varepsilon(v)}{v}}.$$

Also, for all sufficiently large v using (6.37) and (6.38) we get (cf. [B, Eq. (8.32)])

$$(6.43) \quad \begin{aligned} N_f(\tilde{r}(v), t(v)) &\geq \frac{\ln \left(\frac{M_f \left(\frac{\tilde{r}(v)}{t(v)} \right)}{\sqrt{e} M_f(1)} \right)}{k(t(v), \tilde{r}(v))} \geq \frac{(\psi(v - 2\tilde{s}(v)))^{\frac{1}{\kappa(v-2\tilde{s}(v))}} - \phi_f(0) - \frac{1}{2}}{k(t(v), \tilde{r}(v))} \\ &\geq \frac{\left(\frac{m_f(\tilde{r}(v)t(v))}{e} \right)^{\frac{\kappa(v)}{\kappa(v-2\tilde{s}(v))}} - \phi_f(0) - \frac{1}{2}}{k(t(v), \tilde{r}(v))} \geq \frac{(m_f(\tilde{r}(v)t(v)))^{1-\kappa'(v)}}{k(t(v), \tilde{r}(v))}, \end{aligned}$$

where κ' is a positive continuous function in v tending to 0 as $v \rightarrow \infty$.

Equations (6.41)–(6.43) imply that for all sufficiently large $v \geq v_0$,

$$\ln \left(\frac{1+t(v)}{2\sqrt{t(v)}} \right) N_f(\tilde{r}(v), t(v)) \geq (m_f(\tilde{r}(v)t(v)))^{1-\delta(v)} =: k(v)$$

for some positive continuous function $\delta(v)$ tending to 0 as $v \rightarrow \infty$.

Thus, we can proceed as in the proof of Theorem 2.8 of [B] observing that (6.39) gives estimates for a_1 and a_2 similar to those of the theorem (cf. [B, Eq. (8.29)]). \square

6.2. Proof of Theorem 1.7. For $m = 1$ and P, Q the identity automorphisms the required result follows from Theorem A (b) of the previous section by repeating word-for-word the arguments of the proof of Theorem 1.5 in section 5 (cf. Lemma 5.1).

Next, if $f_j : \mathbb{C}^{n_j} \rightarrow \mathbb{C}$, $1 \leq j \leq m$, are of class \mathcal{C} , then due to Theorem 1.1 (f) and the above considered case (of $m = 1$ and P, Q the identity automorphisms), the graph of $F := (f_1 \times \cdots \times f_m) : \mathbb{C}^{\bar{n}} \rightarrow \mathbb{C}^m$, $\bar{n} := n_1 + \cdots + n_m$, satisfies the Bernstein type inequality of exponent $\mu(k) = k^{2+\varepsilon(k)}$, $k \in \mathbb{Z}_+$, with ε as in the statement of the theorem. So that for general P, Q the required result follows from the arguments similar to those of Lemma 4.1.

Specifically, let $s, t \in \mathbb{N}$ be such that all coordinates of holomorphic maps $P^{\pm 1}$ and $Q^{\pm 1}$ belong to $\mathcal{P}_s(\mathbb{C}^{\bar{n}})$ and $\mathcal{P}_t(\mathbb{C}^m)$, respectively. Then the correspondence $h(z, w) \mapsto h(P^{-1}(z), Q(w))$, $z \in \mathbb{C}^{\bar{n}}$, $w \in \mathbb{C}^m$, determines a linear injective map $I : \mathcal{P}_k(\mathbb{C}^{\bar{n}+m}) \rightarrow \mathcal{P}_{k \max(s, t)}(\mathbb{C}^{\bar{n}+m})$. We set $K := P(\text{cl}(\mathbb{B}^{\bar{n}}))$. Since P is a holomorphic automorphism of $\mathbb{C}^{\bar{n}}$, K is a nonpluripolar compact subset of $\mathbb{C}^{\bar{n}}$. Then as in the proof of Lemma 4.1 for $\mu(k) := k^{2+\varepsilon(k)}$, $k \in \mathbb{Z}_+$, with ε as in the statement of Theorem 1.7 and $F_{P, Q} := Q \circ F \circ P$ we obtain

$$\begin{aligned}
& u_{\text{cl}(\mathbb{B}^{\bar{n}}), \mu}^k(z; F_{P, Q}) \\
&= \sup \left\{ \frac{\ln |h_{F_{P, Q}}(z)|}{\max(1, k^{2+\varepsilon(k)})} : h \in \mathcal{P}(\mathbb{C}^{\bar{n}+m}), \deg h = k, \sup_{\mathbb{B}^{\bar{n}}} |h_{F_{P, Q}}| = 1 \right\} \\
&= \sup \left\{ \frac{\ln |I(h)_F(P(z))|}{\max(1, k^{2+\varepsilon(k)})} : h \in \mathcal{P}(\mathbb{C}^{\bar{n}+m}), \deg h = k, \sup_K |I(h)_F| = 1 \right\} \\
&\leq \sup \left\{ \frac{\ln |g_F(P(z))|}{\max(1, k^{2+\varepsilon(k)})} : g \in \mathcal{P}(\mathbb{C}^{\bar{n}+m}), \deg g = k \max(s, t), \sup_K |g_F| = 1 \right\} \\
&\leq \frac{\max(1, (k \max(s, t))^{2+\varepsilon(k \max(s, t))})}{\max(1, k^{2+\varepsilon(k)})} u_{K, \mu}^{k \max(s, t)}(P(z); F) \\
&\leq (\max(s, t))^{2+\varepsilon(0)} u_{K, \mu}^{k \max(s, t)}(P(z); F), \quad z \in \mathbb{C}^{\bar{n}}.
\end{aligned}$$

From here as in (4.26) we get

$$u_{\text{cl}(\mathbb{B}^{\bar{n}}), \mu}(z; F_{P, Q}) \leq (\max(s, t))^{2+\varepsilon(0)} u_{K, \mu}(P(z); F), \quad z \in \mathbb{C}^{\bar{n}}.$$

Since the function $u_{K, \mu}(\cdot; F)$ is locally bounded from above by Theorem 1.2 (b), the previous inequality implies that the function $u_{\text{cl}(\mathbb{B}^{\bar{n}}), \mu}(\cdot; F_{P, Q})$ is locally bounded from above as well. So by Theorem 1.2 (b) the graph $\Gamma_{F_{P, Q}}$ admits the Bernstein type inequality of exponent μ .

This completes the proof of the first statement of the theorem.

Next, if all $n_j = 1$ and $\rho_{f_j} < \infty$, then in the above arguments $\varepsilon = 0$. In this case, $\Gamma_{F_{P,Q}}$ admits the Bernstein type inequality of exponent μ_{id}^2 . This exponent is optimal due to Theorem 1.1 (c).

The proof of the theorem is complete. \square

6.3. Proof of Theorem 1.8. Without loss of generality we may assume that for some $p \in \{1, \dots, m\}$, $\rho_{f_j} < \infty$ if $1 \leq j \leq p$ and $\rho_{f_j} = \infty$ if $p+1 \leq j \leq m$.

For $g \in \mathcal{P}_k(\mathbb{C}^{m+1})$ and $1 \leq j \leq m-1$ we define

$$g_j(z, \mathbf{z}_j) := g(z, f_1(z), \dots, f_j(z), \mathbf{z}_j), \quad z \in \mathbb{C}, \quad \mathbf{z}_j := (z_{j+1}, \dots, z_m) \in \mathbb{C}^{m-j}.$$

Also, we set

$$g_0(z, \mathbf{z}_0) := g(z, \mathbf{z}_0) \quad \text{and} \quad g_m(z) := g(z, f_1(z), \dots, f_m(z)), \\ z \in \mathbb{C}, \quad \mathbf{z}_0 := (z_1, \dots, z_m) \in \mathbb{C}^m.$$

By definition, g_j is an entire function on \mathbb{C}^{m-j+1} such that for each fixed $z \in \mathbb{C}$ function $g_j(z, \cdot) \in \mathcal{P}_k(\mathbb{C}^{m-j})$. In what follows, we add index f_j to all characteristics of Theorem A of section 6.1 related to the function $f := f_j$ (e.g., $r := r_{f_j}$, $\varepsilon := \varepsilon_{f_j}$, etc).

Theorem 1.8 is the direct consequence of the following result.

Theorem 6.2. *There exist numbers $C_j \in \mathbb{R}_+$, $k_j \in \mathbb{N}$ and converging to zero sequences $\{\varepsilon_j(k)\}_{k \geq k_j} \subset \mathbb{R}_+$, $0 \leq j \leq m$, such that $\varepsilon_j = 0$ for $0 \leq j \leq p$ and for all $g \in \mathcal{P}_k(\mathbb{C}^{m+1})$ with $k \geq k_j$ and all $\mathbf{z}_j \in \mathbb{C}^{m-j}$*

$$g_j(\cdot, \mathbf{z}_j) \in \mathcal{F}_{p_j(k), k}(er_{f_{j+1}}(p_j(k)); e; \infty), \quad \text{where} \quad p_j(k) := C_j k^{2j + \varepsilon_j(k)}, \quad 0 \leq j \leq m$$

(here we set $r_{f_{m+1}} := r_{f_m}$).

Proof. We define $C_0 = 1$, $k_0 = 1$, $\varepsilon_0 = 0$ and prove the result by induction on j .

For $j = 0$ the function $g_0 := g \in \mathcal{P}_k(\mathbb{C}^{m+p+1})$. In particular, $g_0 \in \mathcal{F}_{k, k}(er; e; \infty)$ for all positive numbers r and, hence, for $r = r_{f_1}(k)$. This establishes the base of induction.

Next, assuming that the result holds for $0 < j < m$, let us prove it for $j+1$.

To this end, we apply Theorem A (b) to functions $f := f_{j+1}$ and $g(z, w) := g_j(z, w, \mathbf{z}_{j+1})$, $(z, w) \in \mathbb{C}^2$, with $p = k$ equal to $p_j(k)$. Then, by the induction hypothesis, one derives from the theorem that for all $g \in \mathcal{P}_k(\mathbb{C}^{m+1})$ with k such that $p_j(k) \geq \tilde{k}_{j+1} := \max(k_{0f_{j+1}}, r_{f_{j+1}}^{-1}(r_{0f_{j+1}}))$,

$$(6.44) \quad g_{j+1}(\cdot, \mathbf{z}_{j+1}) \in \mathcal{F}_{\tilde{p}_{j+1}(k), k}(r_{f_{j+1}}(p_j(k)); e; \infty), \quad \text{where} \\ \tilde{p}_{j+1}(k) := C_{f_{j+1}}(p_j(k))^{2 + \varepsilon_{f_{j+1}}(p_j(k))}.$$

Next, by the definitions of $p_j(k)$ and $\varepsilon_{f_{j+1}}$,

$$\tilde{p}_{j+1}(k) = C_{f_{j+1}} C_j^{2 + \varepsilon_{f_{j+1}}(p_j(k))} k^{2j+1 + 2\varepsilon_{f_{j+1}}(p_j(k)) + \varepsilon_j(k)(2 + \varepsilon_{f_{j+1}}(p_j(k)))} \\ \leq C_{j+1} k^{2j+1 + \varepsilon_{j+1}(k)} =: p_{j+1}(k).$$

Here

$$\varepsilon_{j+1}(k) := 2^j \varepsilon_{f_{j+1}}(p_j(k)) + \varepsilon_j(k)(2 + \varepsilon_{f_{j+1}}(p_j(k))) \quad \text{and}$$

$$C_{j+1} := \sup_{k \geq \tilde{k}_{j+1}} \left\{ C_{f_{j+1}} C_j^{2+\varepsilon_{f_{j+1}}(p_j(k))} \right\}.$$

(The number is finite because $\varepsilon_{f_{j+1}}$ is a bounded function.)

Note that the above expression for ε_{j+1} and statement (1) of Theorem A (applied to functions f_i for $1 \leq i \leq m$) show that $\varepsilon_{j+1} = 0$ whenever $j+1 \leq m$. For other indices, $\lim_{k \rightarrow \infty} \varepsilon_{j+1}(k) = 0$ (as ε_j and $\varepsilon_{f_{j+1}}$ possess this property and $p_j(k) \rightarrow \infty$ as $k \rightarrow \infty$).

To complete the proof of the inductive step we must show that for all sufficiently large integers k and all $j+2 \leq m$

$$(6.45) \quad er_{f_{j+2}}(p_{j+1}(k)) \leq r_{f_{j+1}}(p_j(k)).$$

To establish this fact we consider three cases.

(1) $j+2 \leq p$. In this case f_{j+1} and f_{j+2} satisfy condition (I), see (1.7). Also, due to equation (6.33), see [B, Eq. (9.3), (9.7)], functions r_{f_s} are right inverses of nondecreasing functions

$$k_{f_s}(r) := \frac{m_{f_s}(e^{-\alpha_{\rho_{f_s}} r}) - m_{f_s}(e^{-2\alpha_{\rho_{f_s}} r}) - 1}{9(\sqrt{e} + 1)^2(\rho_{f_s}^2 + 1)(17 + 2\ln(\rho_{f_s} + 1))}, \quad r \geq r_{0f_s}, \quad s = j+1, j+2.$$

Since $\varepsilon_s = 0$, by the definition of $p_s(k)$, $s = j+1, j+2$, for all sufficiently large integers k ,

$$er_{f_{j+2}}(p_{j+1}(k)) = er_{f_{j+2}}(C_{j+1}k^{2^{j+1}}) \quad \text{and} \quad r_{f_{j+1}}(p_j(k)) = r_{f_{j+1}}(C_jk^{2^j}).$$

Passing here to inverse functions we reduce (6.45) to the question on the validity, for all sufficiently large r , of the inequality

$$\left(\frac{k_{f_{j+2}}\left(\frac{r}{e}\right)}{C_{j+1}} \right)^{\frac{1}{2^{j+1}}} \geq \left(\frac{k_{f_{j+1}}(r)}{C_j} \right)^{\frac{1}{2^j}}.$$

In turn, the latter inequality is the consequence of the following result.

Lemma 6.3. *Under the hypotheses of Theorem 1.8, see (1.9),*

$$\lim_{r \rightarrow \infty} \frac{(k_{f_{j+1}}(r))^2}{k_{f_{j+2}}\left(\frac{r}{e}\right)} = 0.$$

Proof. Making use of explicit expressions for functions k_{f_s} we obtain

$$(6.46) \quad \lim_{r \rightarrow \infty} \frac{(k_{f_{j+1}}(r))^2}{k_{f_{j+2}}\left(\frac{r}{e}\right)} = \lim_{r \rightarrow \infty} \frac{(m_{f_{j+1}}(e^{-\alpha_{\rho_{f_{j+1}}} r}) - m_{f_{j+1}}(e^{-2\alpha_{\rho_{f_{j+1}}} r}))^2}{m_{f_{j+2}}(e^{-\alpha_{\rho_{f_{j+2}}} r}) - m_{f_{j+2}}(e^{-2\alpha_{\rho_{f_{j+2}}} r})}.$$

Since $\alpha_{\rho_f} \leq 1$, see (6.33), by the maximum principle for subharmonic functions

$$(6.47) \quad m_{f_{j+1}}(e^{-\alpha_{\rho_{f_{j+1}}} r}) - m_{f_{j+1}}(e^{-2\alpha_{\rho_{f_{j+1}}} r}) \leq m_{f_{j+1}}(r) - m_{f_{j+1}}\left(\frac{r}{e}\right).$$

Next, assume that $f_{j+2} \in \mathcal{C}$ and the limit superior in equation (1.7) for f_{j+2} is bounded from above by a constant C . Then due to (6.36) for all sufficiently large r ,

$$(6.48) \quad \frac{m_{f_{j+2}}(e^{\alpha_{\rho} f_{j+2}} r) - m_{f_{j+2}}(e^{-\alpha_{\rho} f_{j+2}} r)}{m_{f_{j+2}}(e^{-\alpha_{\rho} f_{j+2}} r) - m_{f_{j+2}}(e^{-2\alpha_{\rho} f_{j+2}} r)} < A := \frac{C^2}{2}.$$

Applying inequality (6.48) $\ell+1$ times, $\ell := \lceil \frac{1}{\alpha_{\rho} f_{j+2}} \rceil$, and after that the maximum principle for subharmonic functions, for all sufficiently large r we obtain

$$(6.49) \quad \begin{aligned} m_{f_{j+2}}(e^{-\alpha_{\rho} f_{j+2}^{-1}} r) - m_{f_{j+2}}(e^{-2\alpha_{\rho} f_{j+2}^{-1}} r) &> \frac{m_{f_{j+2}}(e^{-1} r) - m_{f_{j+2}}(e^{-\alpha_{\rho} f_{j+2}^{-1}} r)}{A} \\ &> \frac{m_{f_{j+2}}(e^{\alpha_{\rho} f_{j+2}^{-1}} r) - m_{f_{j+2}}(e^{-1} r)}{A^2} > \dots > \frac{m_{f_{j+2}}(e^{\ell \alpha_{\rho} f_{j+2}^{-1}} r) - m_{f_{j+2}}(e^{(\ell-1) \alpha_{\rho} f_{j+2}^{-1}} r)}{A^{\ell+1}} \\ &\geq \frac{m_{f_{j+2}}(r) - m_{f_{j+2}}(\frac{r}{e})}{A^{\ell+1}}. \end{aligned}$$

Using inequalities (6.47), (6.49) in the right-hand side of (6.46), due to condition (1.9) of the theorem, we get

$$\lim_{r \rightarrow \infty} \frac{(k_{f_{j+1}}(r))^2}{k_{f_{j+2}}(\frac{r}{e})} \leq \lim_{r \rightarrow \infty} \frac{A^{\ell+1} \cdot (m_{f_{j+1}}(r) - m_{f_{j+1}}(\frac{r}{e}))^2}{m_{f_{j+2}}(r) - m_{f_{j+2}}(\frac{r}{e})} = 0.$$

The proof of the lemma is complete. \square

As we explained earlier, Lemma 6.3 implies inequality (6.45) for all sufficiently large integers k . Now, as $k_{j+1} \in \mathbb{N}$ in the theorem we choose a natural number such that $p_j(k_{j+1}) \geq \tilde{k}_{j+1}$ and that (6.45) is valid for all integers $k \geq k_{j+1}$.

This completes the proof of the inductive step in case (1).

(2) $j+1 = p$. In this case f_{j+1} satisfies condition (I) and f_{j+2} satisfies condition (II), see (1.7), (1.8). Thus, as before, for all sufficiently large k , $r_{f_{j+1}}(p_j(k)) = r_{f_{j+1}}(C_j k^{2^j})$ and, due to Theorem A part (2),

$$r_{f_{j+2}}(p_{j+1}(k)) = \frac{1}{e^2} m_{f_{j+2}}^{-1} \left(\left(C_{j+1} k^{2^{j+1}} \right)^{1+\varepsilon''_{f_{j+2}}(C_{j+1} k^{2^{j+1}})} \right),$$

where the nonnegative function $\varepsilon''_{f_{j+2}}$ decreases to zero.

Next, by the definition of function $k_{f_{j+1}}$, for all sufficiently large r ,

$$k_{f_{j+1}}(r) \leq m_{f_{j+1}}(r).$$

Passing here to right inverse functions we get, for all sufficiently large k ,

$$r_{f_{j+1}}(k) \geq m_{f_{j+1}}^{-1}(k).$$

Using these facts we conclude that in order to establish inequality (6.45) in this case, it suffices to prove that for all sufficiently large k

$$m_{f_{j+2}}^{-1} \left(\left(C_{j+1} k^{2^{j+1}} \right)^2 \right) \leq m_{f_{j+1}}^{-1} (C_j k^{2^j}).$$

The latter can be derived from the following result by passing to inverse functions.

Lemma 6.4. *For all sufficiently large r ,*

$$\left(\frac{1}{C_j} m_{f_{j+1}}(r) \right)^{2^{-j}} \leq \left(\frac{1}{C_{j+1}^2} m_{f_{j+2}}(r) \right)^{2^{-j-2}}.$$

Proof. We apply condition (1.8) for f_{j+2} assigning index $j+2$ to all functions which appear there. According to this condition, for each $\varepsilon > 0$ there exists some $t_\varepsilon > 0$ such that for all $t \geq t_\varepsilon$,

$$-\left(\frac{1}{\ln \psi_{j+2}(t)} \right)' < \frac{\varepsilon}{t^2}.$$

(The minus sign reflects the fact that the derivative of the function is nonpositive.)

Integrating this inequality from t to infinity we get

$$\frac{1}{\ln \psi_{j+2}(t)} < \frac{\varepsilon}{t}, \quad t \geq t_\varepsilon.$$

Due to condition (II) for f_{j+2} this implies, for all sufficiently large t ,

$$\ln m_{f_{j+2}}(e^t) \geq \frac{\ln \psi_{j+2}(t)}{2} > \frac{t}{2\varepsilon}.$$

Let us choose here $\varepsilon := \frac{1}{8(\rho_{f_{j+1}}+1)}$. Then from the previous inequality and the fact that f_{j+1} is of finite order $\rho_{f_{j+1}}$ we obtain

$$\varliminf_{r \rightarrow \infty} \frac{\ln m_{f_{j+1}}(r)}{\ln r} = \rho_{f_{j+1}} < \rho_{f_{j+1}} + 1 = \frac{1}{8\varepsilon} \leq \varliminf_{r \rightarrow \infty} \frac{\ln m_{f_{j+2}}(r)}{4 \ln r}.$$

This implies the required statement of the lemma. \square

Now, choosing $k_{j+1} \in \mathbb{N}$ as at the end of the proof of case (1) we complete the proof of the inductive step in case (2).

(3) $p < j+1$. In this case f_{j+1} and f_{j+2} satisfy condition (II), see (1.8). Thus, as before, for all sufficiently large k and $s = j+1, j+2$,

$$r_{f_s}(p_{s-1}(k)) = \frac{1}{e^2} m_{f_s}^{-1} \left(\left(C_{s-1} k^{2^{s-1}} \right)^{1+\varepsilon''_{f_s}(C_{s-1} k^{2^{s-1}})} \right),$$

where the nonnegative functions ε''_{f_s} decrease to zero.

Hence, to establish inequality (6.45) in this case we must show that for all sufficiently large integers k ,

$$e m_{f_{j+2}}^{-1} \left(\left(C_{j+1} k^{2^{j+1}} \right)^{1+\varepsilon''_{f_{j+2}}(C_{j+1} k^{2^{j+1}})} \right) \leq m_{f_{j+1}}^{-1} \left(\left(C_j k^{2^j} \right)^{1+\varepsilon''_{f_{j+1}}(C_j k^{2^j})} \right).$$

This inequality follows straightforwardly from the next result.

Lemma 6.5. *There exists some $\alpha > 1$ such that for all sufficiently large k ,*

$$e m_{f_{j+2}}^{-1} \left(\left(C_{j+1} k^{2^{j+1}} \right)^\alpha \right) \leq m_{f_{j+1}}^{-1} \left(C_j k^{2^j} \right).$$

Proof. Passing to inverse functions we rewrite the required inequality as the inequality

$$\left(\frac{1}{C_j} m_{f_{j+1}}(r) \right)^{2^{-j}} \leq \left(\frac{1}{C_{j+1}^\alpha} m_{f_{j+2}} \left(\frac{r}{e} \right) \right)^{\frac{2^{-j-1}}{\alpha}}$$

valid for all sufficiently large $r > 0$.

This is true if

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln m_{f_{j+1}}(r)}{\ln m_{f_{j+2}} \left(\frac{r}{e} \right)} < \frac{1}{2\alpha}.$$

But according to condition (1.10) of the theorem

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln m_{f_{j+1}}(r)}{\ln m_{f_{j+2}} \left(\frac{r}{e} \right)} = L < \frac{1}{2}.$$

Hence, it suffices to choose

$$\alpha = \frac{2}{1 + 2L}.$$

This completes the proof of the lemma. \square

Finally, choosing $k_{j+1} \in \mathbb{N}$ as in cases (1), (2) we complete the proof of the inductive step.

Thus Theorem 6.2 is proved by induction on j . \square

Applying Theorem 6.2 with $j = m$, by the definition of class $\mathcal{F}_{p_m(k),k}(er_{f_m}(p_m(k); e; \infty))$ repeating the arguments of the proof of Theorem 1.5 (see Lemma 5.1) we obtain that Γ_f admits the Bernstein type inequality of exponent $\mu(k) := k^{2^m + \varepsilon(k)}$, $k \in \mathbb{Z}_+$, for some $\varepsilon : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ decreasing to zero.

This completes the proof of Theorem 1.8. \square

Remark 6.6. Arguing as in the proof of [B, Th.2.3], one deduces directly from Theorem 6.2 by means of Theorem A (a) that there exist some constant $C \in \mathbb{R}_+$, a number $k_* \in \mathbb{N}$ and a decreasing to 0 continuous function $\varepsilon_* : (k_*, \infty) \rightarrow \mathbb{R}_+$ equal to 0 if all $\rho_{f_j} = 0$ such that for all $p \in \mathcal{P}_k(\mathbb{C}^{m+1})$ with $k \geq k_*$

$$(6.50) \quad \max_{\mathbb{D}^{m+1}} |p| \leq e^{C k^{2^m + \varepsilon_*(k)} \ln(r_{f_1}(p_0(k)) \cdots r_{f_m}(p_{m-1}(k)))} \max_{\mathbb{D}} |p_f|;$$

recall that $p_f(z) := p(z, f_1(z), \dots, f_m(z))$, $z \in \mathbb{C}$, also, $\mathbb{D}^{m+1} := \times^{m+1} \mathbb{D}$.

Here, according to (6.45), for a sufficiently large k_* and all $k \geq k_*$

$$\ln(r_{f_1}(p_0(k)) \cdots r_{f_m}(p_{m-1}(k))) \leq m \ln(r_{f_1}(k)).$$

Moreover, cf. [B, Th. 2.8], for all such k ,

$$r_{f_1}(k) \leq \begin{cases} c_{f_1} k^{\frac{1}{\rho_{f_1}}} & \text{if } 0 < \rho_{f_1} < \infty \\ k^{\delta_{f_1}(k)} & \text{if } 0 < \rho_{f_1} = \infty \end{cases}$$

for a constant $c_{f_1} \in \mathbb{R}_+$ and a decreasing to 0 nonnegative function $\delta_{f_1} \in C([k_*, \infty))$.

7. PROOFS OF PROPOSITIONS 1.11, 1.12 AND 1.14

7.1. Proof of Proposition 1.11. (1) For $\rho_f < \infty$ the assumption of the proposition implies, for some constant $c > 0$,

$$(7.51) \quad \phi_f(t) - c \leq \phi_g(t) \leq \phi_f(t) + c, \quad t \in \mathbb{R}.$$

Hence,

$$\overline{\lim}_{t \rightarrow \infty} \frac{\phi_g(t+1) - \phi_g(t)}{\phi_g(t) - \phi_g(t-1)} \leq \overline{\lim}_{t \rightarrow \infty} \frac{\phi_f(t+1) - \phi_f(t) + 2c}{\phi_f(t) - \phi_f(t-1) - 2c} = \overline{\lim}_{t \rightarrow \infty} \frac{\phi_f(t+1) - \phi_f(t)}{\phi_f(t) - \phi_f(t-1)} < \infty,$$

i.e. $g \in \mathcal{C}$. (Here the last equality is due to the fact that $\phi_f(t+1) - \phi_f(t)$, $t \in \mathbb{R}$, is a nondecreasing unbounded from above function as f is nonpolynomial.)

For $\rho_f = \infty$, inequality (7.51) yields

$$\lim_{t \rightarrow \infty} \frac{\ln \phi_g(t)}{\ln \phi_f(t)} = 1.$$

Thus, conditions (1.8) for f and g coincide, i.e. $g \in \mathcal{C}$.

(2) The assumption of the proposition leads to the inequality:

$$\frac{1}{c} \phi_f(t) \leq \phi_g(t) \leq c \phi_f(t), \quad t \in \mathbb{R},$$

for some constant $c > 1$.

In turn, this implies

$$\lim_{t \rightarrow \infty} \frac{\ln \phi_g(t)}{\ln \phi_f(t)} = 1.$$

Thus, as above, conditions (1.8) for f and g coincide, i.e. $g \in \mathcal{C}$.

(3) The statement holds true because $m_{f^n} = n m_f$ for all $n \in \mathbb{N}$. □

7.2. Proof of Proposition 1.12. Let

$$\lim_{r \rightarrow \infty} \frac{m_f(r)}{r^{\rho(r)}} = \mu_f > 0.$$

Recall that the proximate order satisfies the following property, see, e.g., [L]: *for every $\varepsilon > 0$ and every $0 < a < b < \infty$ there exists $r_0 > 0$ such that for all $k \in [a, b]$ and $r \geq r_0$,*

$$(7.52) \quad (1 - \varepsilon)k^{\rho_f} r^{\rho(r)} < (kr)^{\rho(kr)} < (1 + \varepsilon)k^{\rho_f} r^{\rho(r)}.$$

Let us fix some $\varepsilon \in (0, \min(\frac{1}{2}, \frac{\mu_f}{2}))$ and define a number $d > 0$ from the identity

$$(7.53) \quad \frac{\mu_f}{2} - 3\sigma_f e^{-\rho_f d} = \frac{9\sigma_f - \mu_f e^{-\rho_f}}{4} \cdot \frac{2\mu_f}{9\sigma_f}.$$

Then, due to (7.52), convexity of ϕ_f and definitions of μ_f and σ_f , there exists some $t_\varepsilon \in \mathbb{R}$ such that for all $t \geq t_\varepsilon$,

$$\begin{aligned} \phi_f(t+1) - \phi_f(t) &\leq (\sigma_f + \varepsilon)e^{(t+1)\rho(e^{t+1})} - (\mu_f - \varepsilon)e^{t\rho(e^t)} \\ &\leq (\sigma_f + \varepsilon)(1 + \varepsilon)e^{2\rho_f} e^{(t-1)\rho(e^{t-1})} - (\mu_f - \varepsilon)(1 - \varepsilon)e^{\rho_f} e^{(t-1)\rho(e^{t-1})} \\ &\leq \frac{9\sigma_f e^{\rho_f} - \mu_f}{4} e^{(t-1)\rho(e^{t-1})} e^{\rho_f} = \frac{9\sigma_f e^{2\rho_f} d}{2\mu_f} \cdot \frac{\frac{\mu_f}{2} - 3\sigma_f e^{-d\rho_f}}{d} e^{(t-1)\rho(e^{t-1})} \\ &\leq \frac{9\sigma_f e^{2\rho_f} d}{2\mu_f} \cdot \frac{(\mu_f - \varepsilon)e^{(t-1)\rho(e^{t-1})} - \frac{\sigma_f + \varepsilon}{1 - \varepsilon} e^{-d\rho_f} e^{(t-1)\rho(e^{t-1})}}{d} \\ &\leq \frac{9\sigma_f e^{2\rho_f} d}{2\mu_f} \cdot \frac{(\mu_f - \varepsilon)e^{(t-1)\rho(e^{t-1})} - (\sigma_f + \varepsilon)e^{(t-1-d)\rho(e^{t-1-d})}}{d} \\ &\leq \frac{9\sigma_f e^{2\rho_f} d}{2\mu_f} \cdot \frac{\phi_f(t-1) - \phi_f(t-1-d)}{d} \leq \frac{9\sigma_f e^{2\rho_f} d}{2\mu_f} (\phi_f(t) - \phi_f(t-1)). \end{aligned}$$

This shows that f satisfies (1.7), i.e. $f \in \mathcal{C}$. \square

7.3. Proof of Proposition 1.14. We set $u := \operatorname{Re} f$ and $h := e^f$. Then $M_h(r) = M_{e^u}(r)$ and so $m_h(r) = M_u(r)$, $r > 0$. To prove that $h \in \mathcal{C}$, we consider two cases.

First, assume that $f \in \mathcal{C}$ satisfies conditions (I), see (1.7), and (1.11). We prove that $h \in \mathcal{C}$ satisfies condition (II) with $\psi(t) = \phi_h(t) (= m_h(e^t) = M_u(e^t))$, see (1.8):

Lemma 7.1.

$$\lim_{t \rightarrow \infty} t^2 \left(\frac{1}{m_u(e^t)} \right)' = 0.$$

Proof. Applying the Borel-Carathéodory theorem to f restricted to each complex line passing through the origin we obtain, for $0 < s < 1$ and all $r > 0$,

$$(7.54) \quad M_f(sr) \leq \frac{2}{(1-s)} M_u(r) + \frac{1+s}{1-s} |f(0)|.$$

On the other hand, obviously

$$(7.55) \quad M_u(r) \leq M_f(r).$$

Further, condition (1.7) for f implies, for some $A > 0$ and all sufficiently large t ,

$$\phi_f(t+1) - \phi_f(t) < A (\phi_f(t) - \phi_f(t-1)) < A \phi_f(t).$$

Hence, for such t ,

$$(7.56) \quad \phi_f(t+1) < c_1 \phi_f(t), \quad c_1 := A+1.$$

Thus, for $q(t) := m_u(e^t)$, $t \in \mathbb{R}$, from (7.54) with $s = \frac{1}{e}$, (7.55) and (7.56) we obtain, for all sufficiently large t ,

$$(7.57) \quad q(t+1) \leq \phi_f(t+1) < c_1^2 \phi_f(t-1) \leq c_2 q(t),$$

for some c_2 depending on A and f .

Now, from (7.57) using that q' is a nonnegative nondecreasing function by (1.11) we obtain

$$\begin{aligned} 0 &\leq \liminf_{t \rightarrow \infty} t^2 \left(-\frac{1}{q(t)} \right)' \leq \overline{\lim}_{t \rightarrow \infty} t^2 \left(-\frac{1}{q(t)} \right)' = \overline{\lim}_{t \rightarrow \infty} \frac{t^2 q'(t)}{q^2(t)} \leq \overline{\lim}_{t \rightarrow \infty} \frac{t^2 (q(t+1) - q(t))}{q^2(t)} \\ &\leq \overline{\lim}_{t \rightarrow \infty} \frac{t^2 q(t+1)}{q^2(t)} \leq \overline{\lim}_{t \rightarrow \infty} \frac{t^2 c_2 q(t)}{q^2(t)} \leq \overline{\lim}_{t \rightarrow \infty} \frac{c_2^2 t^2}{\phi_f(t)} = 0. \end{aligned}$$

□

Next, we consider the case of $f \in \mathcal{C}$ satisfying condition (II), see (1.8). Let us prove the following result.

Lemma 7.2. *Function f satisfies condition (1.11).*

Proof. Due to condition (1.8), for each $\varepsilon > 0$ there exists some $t_\varepsilon > 0$ such that for all $t \geq t_\varepsilon$

$$-\left(\frac{1}{\ln \psi(t)} \right)' \leq \frac{\varepsilon}{t^2}.$$

Integrating this inequality from t to ∞ we get for all $t \geq t_\varepsilon$

$$\frac{t}{\ln \psi(t)} \leq \varepsilon.$$

This implies that

$$(7.58) \quad \lim_{t \rightarrow \infty} \frac{t}{\ln \psi(t)} = 0.$$

Further, due to condition (II) there exists some $t_* \in \mathbb{R}$ and a continuous function $\kappa : [t_*, \infty) \rightarrow \mathbb{R}_+$, $\lim_{t \rightarrow \infty} \kappa(t) = 1$, such that

$$(7.59) \quad \ln \psi(t) = \kappa(t) \ln \phi_f(t), \quad t \geq t_*.$$

From here and (7.58) we obtain

$$0 \leq \liminf_{t \rightarrow \infty} \frac{t^2}{\phi_f(t)} \leq \overline{\lim}_{t \rightarrow \infty} \frac{t^2}{\phi_f(t)} \leq \overline{\lim}_{t \rightarrow \infty} \frac{t^2}{(\ln \phi_f(t))^2} = \overline{\lim}_{t \rightarrow \infty} \frac{t^2}{(\ln \psi(t))^2} = 0.$$

That is, f satisfies condition (1.11). □

Due to the lemma, without additional restrictions on f , we must show that e^f satisfies condition (II), see (1.8).

We proceed as in the proof of Theorem A denoting by $s : [t_0, \infty) \rightarrow \mathbb{R}$, for some $t_0 > 0$, a continuous function such that

$$(7.60) \quad \frac{\psi(t)}{\psi(t - 2s(t))} = e.$$

Then for $\tilde{s}(t) := \min(s(t), 1)$,

$$(7.61) \quad \frac{1}{\tilde{s}(t)} \leq (m_f(e^t))^{\frac{\kappa(t)\varepsilon(t)}{t}}, \quad t \geq t_0,$$

where $\varepsilon \in C([t_0, \infty))$ is a positive continuous function tending to zero at ∞ .

Next, according to (7.54), (7.55), for all $t \geq t_0$ for a sufficiently large t_0 ,

$$\phi_f(t - 2\tilde{s}(t)) \leq \ln \frac{\tilde{c}}{\tilde{s}(t)} + m_u(e^t) \quad \text{and} \quad m_u(e^t) \leq \phi_f(t)$$

for an absolute constant $\tilde{c} > 0$. (Recall that $u := \operatorname{Re} f$.)

From here, due to (7.59), (7.60), (7.61) we obtain, for all $t \geq t_0$,

$$(7.62) \quad (1 - \delta(t))(e^{-1}\psi(t))^{\frac{1}{\kappa(t-2\tilde{s}(t))}} \leq (1 - \delta(t))\phi_f(t - 2\tilde{s}(t)) \leq m_u(e^t) \leq \phi_f(t),$$

where $\delta \in C([t_0, \infty))$ is a positive continuous function tending to zero at ∞ .

Now, function $\operatorname{id} - 2\tilde{s} \in C([t_0, \infty))$ tends to ∞ at ∞ and has minimal value in the interval $[t_0 - 2, t_0)$. In particular, each $t \geq t_0$ can be written as $t = v_t - 2\tilde{s}(v_t)$ for some $v_t > t$. So for a sufficiently large $t_0 > 0$ and all $t \geq t_0$ we have by (7.62), (7.60) and (7.61),

$$\begin{aligned} m_u(e^t) &= m_u(e^{v_t - 2\tilde{s}(v_t)}) \geq (1 - \delta(t))(e^{-1}\psi(v_t - 2\tilde{s}(v_t)))^{\frac{1}{\kappa(t-2\tilde{s}(t))}} \\ &\geq (1 - \delta(t))(e^{-2}\psi(v_t))^{\frac{1}{\kappa(t-2\tilde{s}(t))}} = (1 - \delta(t))e^{-\frac{2}{\kappa(t-2\tilde{s}(t))}} (\phi_f(v_t))^{\frac{\kappa(v_t)}{\kappa(t-2\tilde{s}(t))}} \geq (\phi_f(v_t))^{\frac{3}{4}} \\ &\geq (m_u(e^{v_t}))^{\frac{3}{4}} \geq (m_u(e^{v_t}) - m_u(e^{v_t - 2\tilde{s}(v_t)}))^{\frac{3}{4}} \geq (m'_u(e^{v_t - 2\tilde{s}(v_t)}))^{\frac{3}{4}} (2\tilde{s}(v_t))^{\frac{3}{4}} \\ &= (m'_u(e^t))^{\frac{3}{4}} (2\tilde{s}(v_t))^{\frac{3}{4}} \geq (m'_u(e^t))^{\frac{3}{4}} (m_f(e^t))^{-\frac{3\kappa(t)\varepsilon(t)}{4t}} \geq (m'_u(e^t))^{\frac{3}{4}} (m_f(e^t))^{-\frac{1}{4}}. \end{aligned}$$

From the previous inequality and equations (7.62), (7.58), (7.59) we obtain

$$\begin{aligned} 0 &\leq \varliminf_{t \rightarrow \infty} t^2 \left(-\frac{1}{\ln \phi_h(t)} \right)' \leq \varlimsup_{t \rightarrow \infty} t^2 \left(-\frac{1}{\ln \phi_h(t)} \right)' = \varlimsup_{t \rightarrow \infty} t^2 \left(-\frac{1}{m_u(e^t)} \right)' = \varlimsup_{t \rightarrow \infty} \frac{t^2 m'_u(e^t)}{(m_u(e^t))^2} \\ &\leq \varlimsup_{t \rightarrow \infty} \frac{t^2 (m_u(e^t))^{\frac{4}{3}} (\phi_f(t))^{\frac{1}{3}}}{(m_u(e^t))^2} \leq \varlimsup_{t \rightarrow \infty} \frac{t^2 (\psi(t))^{\frac{1}{3\kappa(t)}}}{(1 - \delta(t))^{\frac{2}{3}} (e^{-1}\psi(t))^{\frac{2}{3\kappa(t-2\tilde{s}(t))}}} \\ &\leq \varlimsup_{t \rightarrow \infty} \frac{e^{\frac{2}{3}} t^2}{(\psi(t))^{\frac{2}{3\kappa(t-2\tilde{s}(t))} - \frac{1}{3\kappa(t)}}} \leq \varlimsup_{t \rightarrow \infty} \frac{e^{\frac{2}{3}} t^2}{(\ln \psi(t))^2} = 0. \end{aligned}$$

This shows that $h := e^f$ satisfies condition (1.8), i.e. $h \in \mathcal{C}$.

Further, under the hypotheses of the proposition, we show that $\sin f$ and $\cos f$ are of class \mathcal{C} . In fact, by the definition of the trigonometric functions, for all sufficiently large r ,

$$\frac{1}{3}M_{e^{if}}(r) \leq \frac{1}{2}M_{e^{if}}(r) - \frac{1}{2}M_{e^{-if}}(r) \leq \max(M_{\sin f}(r), M_{\cos f}(r)) \leq M_{e^{if}}(r).$$

Since, as we have proved, $e^{if} \in \mathcal{C}$, the above inequality and Proposition 1.11 (1) imply that $\sin f$ and $\cos f$ are of class \mathcal{C} as well.

To complete the proof of the proposition, it remains to show that if $f \in \mathcal{C}$ is a univariate entire function satisfying condition (1.11), then its derivative and every antiderivative are of class \mathcal{C} and satisfy (1.11).

Note that according to the Cauchy estimates for the derivative of a holomorphic function, for $0 < s < 1$ and all $r > 0$,

$$(7.63) \quad M_{f'}(sr) \leq \frac{1}{1-s} M_f(r).$$

On the other hand, by the mean-value theorem

$$(7.64) \quad M_f(r) - |f(0)| \leq r M_{f'}(r).$$

First, assume that $f \in \mathcal{C}$ satisfies conditions (1.7) and (1.11). Then for some $C > 0$ and all sufficiently large t ,

$$(7.65) \quad \frac{\phi_f(t+1) - \phi_f(t)}{\phi_f(t) - \phi_f(t-1)} < C.$$

Applying (7.63) with $s = e^{-\frac{1}{2}}$, (7.64) and then (7.65) and convexity of ϕ_f we obtain, for all sufficiently large t ,

$$(7.66) \quad \begin{aligned} \frac{\phi_{f'}(t+1) - \phi_{f'}(t)}{\phi_{f'}(t) - \phi_{f'}(t-1)} &\leq \frac{\phi_f(t+\frac{3}{2}) - \phi_f(t) + t + c_1}{\phi_f(t) - \phi_f(t-\frac{1}{2}) - t + c_2} \leq \frac{(C+1)(\phi_f(t+\frac{1}{2}) - \phi_f(t-\frac{1}{2})) + t + c_1}{\frac{1}{2}(\phi_f(t-\frac{1}{2}) - \phi_f(t-\frac{3}{2})) - t + c_2} \\ &\leq \frac{C(C+1)(\phi_f(t-\frac{1}{2}) - \phi_f(t-\frac{3}{2})) + t + c_1}{\frac{1}{2}(\phi_f(t-\frac{1}{2}) - \phi_f(t-\frac{3}{2})) - t + c_2} \end{aligned}$$

for some absolute constants $c_1, c_2 \in \mathbb{R}$.

Further, due to convexity of ϕ_f , for all $t > 0$,

$$\phi_f(t-\frac{1}{2}) - \phi_f(t-\frac{3}{2}) \geq \frac{\phi_f(t-\frac{3}{2}) - \phi_f(0)}{t-\frac{3}{2}}.$$

This and condition (1.11) imply

$$(7.67) \quad \begin{aligned} 0 &\leq \lim_{t \rightarrow \infty} \frac{t}{\phi_f(t-\frac{1}{2}) - \phi_f(t-\frac{3}{2})} \leq \overline{\lim}_{t \rightarrow \infty} \frac{t}{\phi_f(t-\frac{1}{2}) - \phi_f(t-\frac{3}{2})} \leq \lim_{t \rightarrow \infty} \frac{t(t-\frac{3}{2})}{\phi_f(t-\frac{3}{2}) - \phi_f(0)} \\ &= \lim_{t \rightarrow \infty} \frac{(t-\frac{3}{2})^2}{\phi_f(t-\frac{3}{2})} = 0. \end{aligned}$$

Therefore from (7.66), (7.67) we deduce

$$\overline{\lim}_{t \rightarrow \infty} \frac{\phi_{f'}(t+1) - \phi_{f'}(t)}{\phi_{f'}(t) - \phi_{f'}(t-1)} \leq \overline{\lim}_{t \rightarrow \infty} \frac{C(C+1)(\phi_f(t - \frac{1}{2}) - \phi_f(t - \frac{3}{2})) + t + c_1}{\frac{1}{2}(\phi_f(t - \frac{1}{2}) - \phi_f(t - \frac{3}{2})) - t + c_2} \leq 2C(C+1).$$

This is condition (1.7) for f' , i.e. $f' \in \mathcal{C}$.

Let us show that f' satisfies condition (1.11). Indeed, using (7.64) and condition (1.11) for f we get, for all sufficiently large t ,

$$\lim_{t \rightarrow \infty} \frac{\phi_{f'}(t)}{t^2} \geq \lim_{t \rightarrow \infty} \frac{\phi_f(t) - t + c}{t^2} = \lim_{t \rightarrow \infty} \frac{\phi_f(t)}{t^2} = \infty,$$

as required.

Now, let us prove that if $f' \in \mathcal{C}$ and satisfies (1.7), (1.11), then its antiderivative $f \in \mathcal{C}$ and satisfy these conditions as well.

As before, we apply inequalities (7.63), (7.64) and instead of (7.65) we use the condition

$$(7.68) \quad \frac{\phi_{f'}(t+1) - \phi_{f'}(t)}{\phi_{f'}(t) - \phi_{f'}(t-1)} < C.$$

Similarly to (7.66) we derive, for all sufficiently large t , that

$$(7.69) \quad \begin{aligned} \frac{\phi_f(t+1) - \phi_f(t)}{\phi_f(t) - \phi_f(t-1)} &\leq \frac{\phi_{f'}(t+1) - \phi_{f'}(t - \frac{1}{2}) + t + c_3}{\phi_{f'}(t - \frac{1}{2}) - \phi_{f'}(t-1) - t + c_4} \\ &\leq \frac{C(C+1)(\phi_{f'}(t-1) - \phi_{f'}(t-2)) + t + c_3}{\frac{1}{2}(\phi_{f'}(t-1) - \phi_{f'}(t-2)) - t + c_4} \end{aligned}$$

for some absolute constants $c_3, c_4 \in \mathbb{R}$.

From here, as above (cf. (7.67)), we obtain that

$$\overline{\lim}_{t \rightarrow \infty} \frac{\phi_f(t+1) - \phi_f(t)}{\phi_f(t) - \phi_f(t-1)} \leq 2C(C+1),$$

i.e. $f \in \mathcal{C}$.

Also, since f' satisfies (1.11), using (7.63) we get

$$\lim_{t \rightarrow \infty} \frac{\phi_f(t)}{t^2} \geq \lim_{t \rightarrow \infty} \frac{\phi_{f'}(t - \frac{1}{2}) - 1}{t^2} = \lim_{t \rightarrow \infty} \frac{\phi_{f'}(t)}{t^2} = \infty.$$

Hence, f satisfies (1.11) as well and so the required statements of the proposition are proved for functions of finite order.

Next, we consider the case of $f \in \mathcal{C}$ satisfying condition (1.8). We apply inequalities (7.63), (7.64) for $s = e^{-2\tilde{s}(t)}$ with $\tilde{s}(t)$ as in (7.60), (7.61), i.e. $\tilde{s}(t) := \min(s(t), 1)$, where $s : [t_0, \infty) \rightarrow \mathbb{R}$, for some $t_0 > 0$, is a continuous function such that

$$(7.70) \quad \frac{\psi(t)}{\psi(t - 2s(t))} = e.$$

Then we have

$$(7.71) \quad \frac{1}{\tilde{s}(t)} \leq (\phi_f(t))^{\frac{\kappa(t)\varepsilon(t)}{t}}, \quad t \geq t_0,$$

where $\varepsilon \in C([t_0, \infty))$ is a positive continuous function tending to zero at ∞ and κ is determined by (7.59).

As before, each $t \geq t_0$ can be written as $t = v_t - 2\tilde{s}(v_t)$ for some $v_t > t$. Hence, from (7.63), (7.70), (7.69) and (7.59) we obtain, for all sufficiently large t ,

$$(7.72) \quad \begin{aligned} \phi_{f'}(t) &= \phi_{f'}(v_t - 2\tilde{s}(v_t)) \leq \phi_f(v_t) + \ln \left(\frac{1}{\tilde{s}(v_t)} \right) + c \leq 2\phi_f(v_t) = 2(\psi(v_t))^{\frac{1}{\kappa(v_t)}} \\ &\leq 2(e\psi(v_t - 2\tilde{s}(v_t)))^{\frac{1}{\kappa(v_t)}} = 2e^{\frac{1}{\kappa(v_t)}} (\phi_f(t))^{\frac{\kappa(t)}{\kappa(v_t)}} \leq 6(\phi_f(t))^{\tilde{\kappa}(t)}; \end{aligned}$$

here c is an absolute constant and $\kappa(t)$ tends to one as $t \rightarrow \infty$.

Further, from (7.64) we deduce that, for all sufficiently large t ,

$$(7.73) \quad \phi_f(t) \leq t + 1 + \phi_{f'}(t).$$

Equations (7.72), (7.73) imply that

$$\lim_{t \rightarrow \infty} \frac{\ln \phi_f(t)}{\ln \phi_{f'}(t)} = 1.$$

Therefore due to condition (II), see (1.8), function $f \in \mathcal{C}$ if and only if $f' \in \mathcal{C}$.

This completes the proof of the proposition. \square

8. PROOFS OF THEOREM 1.17, PROPOSITION 1.19 AND COROLLARY 1.21

8.1. Proof of Theorem 1.17. First, we show that the the radius of convergence r_f of the Taylor expansion of f at 0 is ∞ , i.e. that f is an entire function. By definition,

$$\ln r_f = \lim_{j \rightarrow \infty} \frac{-\ln |c_j|}{j} = \lim_{j \rightarrow \infty} \frac{1}{j} \int_0^j h^{-1}(s) ds \geq \lim_{j \rightarrow \infty} \frac{1}{j} \int_{\frac{j}{2}}^j h^{-1}(s) ds \geq \lim_{j \rightarrow \infty} \frac{h^{-1}(\frac{j}{2})}{2} = \infty,$$

as required.

Next, we prove that f is of finite order. Observe that the second condition for h , see (1.12), implies that for some $c > 0$ and all sufficiently large $t > 0$, $h(t) \leq e^{ct}$. Passing in this inequality to inverse functions we obtain for all $t \geq t_0$, for some $t_0 \in \mathbb{R}_+$,

$$h^{-1}(t) \geq \frac{\ln t}{c}.$$

From here, by the definition of the order of f , we get

$$\rho_f := \overline{\lim}_{j \rightarrow \infty} \frac{j \ln j}{-\ln |c_j|} = \overline{\lim}_{j \rightarrow \infty} \frac{j \ln j}{\int_0^j h^{-1}(s) ds} \leq \overline{\lim}_{j \rightarrow \infty} \frac{j \ln j}{\int_{t_0}^j h^{-1}(s) ds} \leq \overline{\lim}_{j \rightarrow \infty} \frac{cj \ln j}{\int_{t_0}^j \ln s ds} = c,$$

as required.

Since $\rho_f \leq c$, for all sufficiently large $t > 0$ we have, see, e.g., [L, Ch.I.2],

$$(8.74) \quad \nu_f(t) \leq \phi_f(t) \leq 2ct + \nu_f(t),$$

where

$$\nu_f(t) := \sup_{j \in \mathbb{N}} (\ln |c_j| + jt), \quad t \in \mathbb{R}_+.$$

Hence,

$$(8.75) \quad \begin{aligned} \phi_f(t+1) - \phi_f(t) &\leq \nu_f(t+1) - \nu_f(t) + 2c(t+1); \\ \phi_f(t) - \phi_f(t-1) &\geq \nu_f(t) - \nu_f(t-1) - 2c(t-1). \end{aligned}$$

Let us consider the function

$$g(x, t) = - \int_0^x h^{-1}(s) ds + xt, \quad (x, t) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

One easily checks that for a fixed $t \geq h^{-1}(0)$ the function $g(\cdot, t)$ attains its maximal value at $x = h(t)$. Using the substitution $s \mapsto h(s)$ and then the integration by parts we obtain

$$\begin{aligned} g(h(t), t) &= - \int_0^{h(t)} h^{-1}(s) ds + h(t)t = - \int_{h^{-1}(0)}^{h^{-1}(h(t))} sh'(s) ds + h(t)t \\ &= h(t)(t - h^{-1}(h(t))) + \int_{h^{-1}(0)}^{h^{-1}(h(t))} h(s) ds = \int_{h^{-1}(0)}^t h(s) ds. \end{aligned}$$

In particular, for $t \geq h^{-1}(0)$,

$$g(\lfloor h(t) \rfloor, t) = - \int_0^{\lfloor h(t) \rfloor} h^{-1}(s) ds + \lfloor h(t) \rfloor t \leq \nu_f(t) \leq g(h(t), t) = \int_{h^{-1}(0)}^t h(s) ds.$$

Also, by definition, for such t ,

$$\begin{aligned} 0 &\leq g(h(t), t) - g(\lfloor h(t) \rfloor, t) = \{h(t)\}t - \int_{\lfloor h(t) \rfloor}^{h(t)} h^{-1}(s) ds = \int_{\lfloor h(t) \rfloor}^{h(t)} (h^{-1}(h(t)) - h^{-1}(s)) ds \\ &\leq \omega_{h(t)}(1; h^{-1}) \leq t. \end{aligned}$$

This yields (for all $t \geq h^{-1}(0)$)

$$(8.76) \quad \int_{h^{-1}(0)}^t h(s) ds - t \leq \int_{h^{-1}(0)}^t h(s) ds - \omega_{h(t)}(1; h^{-1}) \leq \nu_f(t) \leq \int_{h^{-1}(0)}^t h(s) ds.$$

Using these estimates in (8.75) we get, for all sufficiently large $t > 0$,

$$(8.77) \quad \phi_f(t+1) - \phi_f(t) \leq \int_t^{t+1} h(s) ds + t + 2c(t+1) \leq h(t+1) + (2c+1)(t+1)$$

and

$$(8.78) \quad \phi_f(t) - \phi_f(t-1) \geq \int_{t-1}^t h(s) ds - t - 2c(t-1) \geq h(t-1) - (2c+1)t.$$

Invoking properties of h , we derive from the last two inequalities that

$$\overline{\lim}_{t \rightarrow \infty} \frac{\phi_f(t+1) - \phi_f(t)}{\phi_f(t) - \phi_f(t-1)} \leq \overline{\lim}_{t \rightarrow \infty} \frac{h(t+1)}{h(t-1)} < \infty.$$

Thus, $f \in \mathcal{C}$.

Also, due to (1.12),

$$\lim_{t \rightarrow \infty} \frac{\phi_f(t)}{t^2} \geq \lim_{t \rightarrow \infty} \frac{\nu_f(t)}{t^2} \geq \lim_{t \rightarrow \infty} \frac{\int_{h^{-1}(0)}^t h(s) ds - t}{t^2} \geq \lim_{t \rightarrow \infty} \frac{\int_{\frac{t}{2}}^t h(s) ds}{t^2} \geq \lim_{t \rightarrow \infty} \frac{h(\frac{t}{2})}{2t} = \infty.$$

Hence, f satisfies condition (1.11).

Finally,

$$\rho_f = \overline{\lim}_{t \rightarrow \infty} \frac{\ln \phi_f(t)}{t} = \overline{\lim}_{t \rightarrow \infty} \frac{\ln \nu_f(t)}{t}.$$

Therefore from (8.76) we obtain

$$\rho_f = \overline{\lim}_{t \rightarrow \infty} \frac{\ln h(t)}{t}.$$

The proof of the theorem is complete. \square

8.2. Proof of Proposition 1.19. Clearly, it suffices to prove that under the hypotheses of the proposition, $f_{h_1} + cf_{h_2} \in \mathcal{C}$ for all $c \in \mathbb{C} \setminus 0$.

Due to [L, Ch. I.2, Eq. (1.10)] (cf. (8.74) above) and (8.76), for each $C > \max(\rho_{f_{h_1}}, \rho_{f_{h_2}})$ there exists $t_C > 0$ such that for all $t \geq t_C$,

$$(8.79) \quad \int_{h_i^{-1}(0)}^t h_i(s) ds - \omega_{h_i(t)}(1; h_i^{-1}) \leq \phi_{f_{h_i}}(t) \leq Ct + \int_{h_i^{-1}(0)}^t h_i(s) ds, \quad i = 1, 2.$$

Also, the assumption of the proposition implies that for some $q > \overline{\lim}_{t \rightarrow \infty} \frac{\omega_t(1; h_1^{-1})}{h_1^{-1}(t)} =: \omega_{h_1}$ and all sufficiently large $t > 0$,

$$(8.80) \quad h_2(t) < h_1(t) - q - \rho_{f_{h_1}}.$$

In particular, we obtain that $\rho_{f_{h_2}} \leq \rho_{f_{h_1}}$, see Theorem 1.17.

Inequality (8.80) shows that for each $\tilde{q} \in (\omega_{h_1}, q)$ there exists $t_{\tilde{q}} > 0$ such that for all $t \geq t_{\tilde{q}}$,

$$\int_{h_2^{-1}(0)}^t h_2(s) ds < \int_{h_1^{-1}(0)}^t h_1(s) ds - (\tilde{q} + \rho_{f_{h_1}})t.$$

From here and (8.79) with $C := \rho_{f_{h_1}} + \frac{\tilde{q} - \omega_{h_1}}{2}$ we obtain, for all sufficiently large $t > 0$,

$$\begin{aligned} \phi_{f_{h_2}}(t) &< \left(C - \tilde{q} - \rho_{f_{h_1}} + \frac{\omega_{h_1(t)}(1; h_1^{-1})}{t} \right) t + \phi_{f_{h_1}}(t) \\ &< \left(-\frac{\tilde{q} + \omega_{h_1}}{2} + \omega_{h_1} + \frac{\tilde{q} - \omega_{h_1}}{4} \right) t + \phi_{f_{h_1}}(t) = \frac{\omega_{h_1} - \tilde{q}}{4} t + \phi_{f_{h_1}} \\ &< -\ln(1 + |c|) + \phi_{f_{h_1}}(t). \end{aligned}$$

Thus for all sufficiently large $r > 0$,

$$M_{cf_{h_2}}(r) < \frac{|c|}{1 + |c|} M_{f_{h_1}}(r).$$

This implies that

$$\frac{1}{1+|c|} \leq \lim_{r \rightarrow \infty} \frac{M_{f_{h_1}+cf_{h_2}}(r)}{M_{f_{h_1}}(r)} \leq \overline{\lim}_{r \rightarrow \infty} \frac{M_{f_{h_1}+cf_{h_2}}(r)}{M_{f_{h_1}}(r)} \leq 2.$$

Therefore $f_{h_1} + cf_{h_2} \in \mathcal{C}$ (cf. property (1) in section 1.4).

The proof of the proposition is complete. \square

8.3. Proof of Corollary 1.21. We use inequalities (8.77), (8.78) for functions h_1, \dots, h_l . Then we have, for a fixed $c > \max\{\rho_{f_{h_1}}, \dots, \rho_{f_{h_l}}\}$ and all sufficiently large $t > 0$,

$$h_j(t-1) - (2c+1)t \leq \phi_{f_{h_j}}(t) - \phi_{f_{h_j}}(t-1) \leq h_j(t+1) + (2c+1)(t+1).$$

Together with conditions (1.12) for functions h_j , $1 \leq j \leq l$, this implies

$$\begin{aligned} \overline{\lim}_{r \rightarrow \infty} \frac{m_{f_j}(r) - m_{f_j}\left(\frac{r}{e}\right)}{\sqrt{m_{f_{j+1}}(r) - m_{f_{j+1}}\left(\frac{r}{e}\right)}} &\geq \overline{\lim}_{t \rightarrow \infty} \frac{h_j(t-1) - (2c+1)t}{\sqrt{h_{j+1}(t+1) + (2c+1)(t+1)}} \geq A^{-\frac{3}{2}} \overline{\lim}_{t \rightarrow \infty} \frac{h_j(t)}{\sqrt{h_{j+1}(t)}}; \\ \overline{\lim}_{r \rightarrow \infty} \frac{m_{f_j}(r) - m_{f_j}\left(\frac{r}{e}\right)}{\sqrt{m_{f_{j+1}}(r) - m_{f_{j+1}}\left(\frac{r}{e}\right)}} &\leq \overline{\lim}_{t \rightarrow \infty} \frac{h_j(t+1) + (2c+1)(t+1)}{\sqrt{h_{j+1}(t-1) - (2c+1)t}} \leq A^{\frac{3}{2}} \overline{\lim}_{t \rightarrow \infty} \frac{h_j(t)}{\sqrt{h_{j+1}(t)}}, \end{aligned}$$

where

$$A := \max_{1 \leq j \leq l} \left\{ \overline{\lim}_{t \rightarrow \infty} \frac{h_j(t+1)}{h_j(t)} \right\}.$$

Hence, condition (1.9) of Theorem 1.8 acquires the form

$$0 = \lim_{r \rightarrow \infty} \frac{m_{f_j}(r) - m_{f_j}\left(\frac{r}{e}\right)}{\sqrt{m_{f_{j+1}}(r) - m_{f_{j+1}}\left(\frac{r}{e}\right)}} = \lim_{t \rightarrow \infty} \frac{h_j(t)}{\sqrt{h_{j+1}(t)}} \quad \text{for all } 1 \leq j \leq l-1.$$

Further, let $u_j := \text{Ref}_{h_j}$, $l+1 \leq j \leq m$. Then $\ln m_{e^{f_{h_j}}}(r) = m_{u_j}(r)$ for all such j . Using the Borel-Carathéodory theorem (cf. (7.54)) we obtain, for $0 < s < 1$ and all $r > 0$,

$$(8.81) \quad m_{h_j}(sr) \leq m_{u_j}(r) - \ln(1-s) + c_j$$

for some constant $c_j := c(h_j)$.

On the other hand,

$$(8.82) \quad m_{u_j}(r) \leq m_{h_j}(r).$$

Applying (8.81) with $s = 1 - e^{-t}$, (8.82) together with (8.74), (8.76) for functions h_j , $l+1 \leq j \leq m$, we obtain, for all sufficiently large $r := e^t$,

(8.83)

$$\begin{aligned} \frac{m_{u_j}(r)}{m_{u_{j+1}}(\frac{r}{e})} &\leq \frac{\phi_{h_j}(t)}{\phi_{h_{j+1}}(t-1+\ln(1-e^{-t}))-t+c_1} \leq \frac{c_2 t + \int_{h_j^{-1}(0)}^t h_j(s) ds}{\int_{h_{j+1}^{-1}(0)}^{t-1-2e^{-t}} h_{j+1}(s) ds - 2t + c_3} \\ \frac{m_{u_j}(r)}{m_{u_{j+1}}(\frac{r}{e})} &\geq \frac{\phi_{h_j}(t+\ln(1-e^{-t}))-t+c_4}{\phi_{h_{j+1}}(t-1)} \geq \frac{\int_{h_j^{-1}(0)}^{t-2e^{-t}} h_j(s) ds - 2t + c_5}{c_6 t + \int_{h_{j+1}^{-1}(0)}^{t-1} h_{j+1}(s) ds}. \end{aligned}$$

for some constants c_1, \dots, c_6 .

To proceed we require

Lemma 8.1. *We have*

$$\lim_{t \rightarrow \infty} \frac{\int_{h_j^{-1}(0)}^{t-2e^{-t}} h_j(s) ds}{\int_{h_j^{-1}(0)}^t h_j(s) ds} = 1 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{\int_{h_{j+1}^{-1}(0)}^{t-1-2e^{-t}} h_{j+1}(s) ds}{\int_{h_{j+1}^{-1}(0)}^{t-1} h_{j+1}(s) ds} = 1.$$

Proof. By the definition of h_j , see (1.12), for all sufficiently large t ,

$$ce^{-t}h_j(t) \leq 2e^{-t}h_j(t-1) \leq \int_{t-2e^{-t}}^t h_j(s) ds \leq 2e^{-t}h_j(t)$$

for some constant $c \in (0, 1)$.

On the other hand,

$$ch_j(t) \leq h_j(t-1) \leq \int_{h_j^{-1}(0)}^t h_j(s) ds \leq th_j(t).$$

Comparing these inequalities, we obtain the first statement of the lemma. The second statement can be proved analogously. \square

Using this lemma together with (8.83) and (1.12) we get, for $l+1 \leq j \leq m-1$,

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln m_{e^{h_j}}(r)}{\ln m_{e^{h_{j+1}}}(r)} = \overline{\lim}_{r \rightarrow \infty} \frac{m_{u_j}(r)}{m_{u_{j+1}}(\frac{r}{e})} = \overline{\lim}_{t \rightarrow \infty} \frac{\int_M^t h_j(s) ds}{\int_M^{t-1} h_{j+1}(s) ds},$$

where $M := \max_{l+1 \leq j \leq m} \{h_j^{-1}(0)\}$.

This expression and Theorem 1.8 give the required statement.

The proof of the corollary is complete. \square

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